

1(a) [3 marks] The log-likelihood is obtained directly from the pdf for t ,

$$\ln L(\tau) = \ln f(t|\tau) = -\ln \tau - \frac{t}{\tau} .$$

Setting the derivative of $\ln L(\tau)$ to zero,

$$\frac{\partial \ln L}{\partial \tau} = -\frac{1}{\tau} + \frac{t}{\tau^2} = 0 ,$$

and solving for τ gives $\hat{\tau} = t$.

1(b-i) [3 marks] For a test of size α , the critical region, $t \geq t_{\text{cut}}$, is determined by the requirement

$$\alpha = P(t \geq t_{\text{cut}}|\tau) = \int_{t_{\text{cut}}}^{\infty} \frac{1}{\tau} e^{-t/\tau} dt = e^{-t_{\text{cut}}/\tau} .$$

Solving for t_{cut} therefore gives $t_{\text{cut}} = -\tau \ln \alpha$.

1(b-ii) [2 marks] Taking larger t values to constitute increasing incompatibility with τ , the p -value is the probability to observe a value greater than or equal to t , i.e.,

$$p = \int_t^{\infty} \frac{1}{\tau} e^{-t'/\tau} dt' = e^{-t/\tau} .$$

1(b-iii) [2 marks] The lower limit at confidence level $1 - \alpha$ is the value of τ for which $p = \alpha$, i.e.,

$$\tau_{\text{lo}} = -\frac{t}{\ln \alpha} = -\frac{1 \text{ s}}{\ln(0.05)} = 0.33 \text{ s} .$$

1(c) [4 marks] The Jeffreys prior is defined by

$$\pi(\tau) \propto \sqrt{I(\tau)} ,$$

where

$$I(\tau) = -E \left[\frac{\partial^2 \ln L}{\partial \tau^2} \right]$$

is the Fisher information. The required ingredients are

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \tau^2} &= \frac{1}{\tau^2} - \frac{2t}{\tau^3}, \\ -E \left[\frac{\partial^2 \ln L}{\partial \tau^2} \right] &= -\frac{1}{\tau^2} + \frac{2E[t]}{\tau^3} = \frac{1}{\tau^2},\end{aligned}$$

where to arrive at the final equality we used the expectation value $E[t] = \tau$. The Jeffreys prior is therefore

$$\pi(\tau) \propto \frac{1}{\tau} \quad (\tau \geq 0).$$

1(d) [3 marks] From Bayes' theorem we find the posterior pdf up to a normalization constant C ,

$$\begin{aligned}p(\tau|t) &\propto f(t|\tau)\pi(\tau) \\ &= C \frac{1}{\tau} e^{-t/\tau} \frac{1}{\tau}.\end{aligned}$$

To determine the constant C we require

$$C \int_0^\infty \frac{1}{\tau^2} e^{-t/\tau} d\tau = 1.$$

To calculate the integral let $u = t/\tau$, $du = (t/u^2)du$, which gives

$$C \int_\infty^0 \left(\frac{u}{t}\right)^2 e^{-u} \left(\frac{-t}{u^2}\right) du = Ct^{-1} = 1,$$

and therefore $C = t$. The normalized posterior pdf is therefore

$$p(\tau|t) = \frac{t}{\tau^2} e^{-t/\tau}.$$

1(e) [3 marks] The posterior mode is found from

$$\frac{\partial p(\tau|t)}{\partial \tau} = \frac{t}{\tau^2} e^{-t/\tau} \left(\frac{t}{\tau^2}\right) + e^{-t/\tau} \left(\frac{-2t}{\tau^3}\right) = 0.$$

Solving for τ gives the mode,

$$\tau_{\text{mode}} = \frac{t}{2}.$$

From Bayes' theorem the posterior is $p(\tau|t) \propto L(\tau)\pi(\tau)$, so if the prior $\pi(\tau)$ is a constant, then the mode of $p(\tau|t)$ is at the same value as the maximum of the likelihood $L(\tau)$, i.e., the posterior mode is then equal to the ML estimator.

The Jeffreys prior, however, is $\pi(\tau) \propto 1/\tau$, so a priori lower values of τ are favoured, and this feature is inherited by the posterior pdf. Therefore the posterior mode is at a lower value ($\tau_{\text{mode}} = t/2$) when using the Jeffreys prior compared to the ML estimator $\hat{\tau}_{\text{ML}} = t$.