Statistical Data Analysis
Problem sheet #9 solutions

1(a) The likelihood function is given by the binomial distribution evaluated with the single observed value n and regarded as a function of the unknown parameter θ:

$$L(\theta) = \frac{N!}{n!(N-n)!}\theta^n (1-\theta)^{N-n} \ .$$

The log-likelihood function is therefore

$$\ln L(\theta) = n \ln \theta + (N-n)\ln(1-\theta) + C \ ,$$

where C represents terms not depending on θ. Setting the derivative of $\ln L$ equal to zero,

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \frac{N-n}{1-\theta} = 0 \ ,$$

we find the ML estimator to be

$$\hat{\theta} = \frac{n}{N} \ .$$

1(b) We are given the expectation and variance of a binomial distributed variable as $E[n] = N\theta$ and $V[n] = N\theta(1-\theta)$. Using these results we find the expectation value of $\hat{\theta}$ to be

$$E[\hat{\theta}] = E\left[\frac{n}{N}\right] = \frac{E[n]}{N} = \frac{N\theta}{N} = \theta \ ,$$

and therefore the bias is $b = E[\hat{\theta}] - \theta = 0$. Similarly we find the variance to be

$$V[\hat{\theta}] = V\left[\frac{n}{N}\right] = \frac{1}{N^2}V[n] = \frac{N\theta(1-\theta)}{N^2} = \frac{\theta(1-\theta)}{N} \ .$$

1(c) Suppose we observe $n = 0$ for $N = 10$ trials. The upper limit on θ at a confidence level of $\mathrm{CL} = 1 - \alpha$ is the value of θ for which there is a probability α to find as few events as we found or fewer, i.e.,

$$\alpha = P(n \leq 0; N, \theta) = \frac{N!}{0!(N-0)!}\theta^0 (1-\theta)^{N-0} \ .$$

Solving for θ gives the 95% CL upper limit

$$\theta_{\mathrm{up}} = 1 - \alpha^{1/N} = 1 - 0.05^{1/10} = 0.26 \ .$$

1(d) To find the Jeffreys prior we need the second derivative of $\ln L$,

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\theta^2} - \frac{N-n}{(1-\theta)^2} \ .$$

The expected Fisher information is therefore

$$I(\theta) = -E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{N\theta}{\theta^2} + \frac{N(1-\theta)}{(1-\theta)^2} = \frac{N}{\theta} + \frac{N}{1-\theta} = \frac{N}{\theta(1-\theta)} \ .$$

The Jeffreys prior is therefore

$$\pi(\theta) \propto \frac{1}{\sqrt{\theta(1-\theta)}} \ .$$

Using this in Bayes theorem to find the posterior pdf gives

$$p(\theta|n) \propto L(n|\theta)\pi(\theta) \propto \frac{\theta^n (1-\theta)^{N-n}}{\sqrt{\theta(1-\theta)}} = \theta^{n-1/2}(1-\theta)^{N-n-1/2} \ .$$

1(e) To find a Bayesian upper limit on θ one simply integrates the posterior pdf so that a specified probability $1-\alpha$ is contained below $\theta_{\rm up}$, i.e.,

$$1 - \alpha = \int_0^{\theta_{\rm up}} p(\theta|n)\, d\theta \ ,$$

solving for $\theta_{\rm up}$ gives the upper limit.

A frequentist upper limit as found in (c) is a function of the data designed to be greater than the true value of the parameter with a fixed probability (the confidence level) regardless of the parameter's actual value. A Bayesian interval can be regarded as reflecting a range for the parameter where it is believed to lie with a fixed probability (the credibility level). Note that with the Jeffreys prior, one may not necessary use the degree of belief interpretation of the interval, but rather take it to have a certain probability to cover the true θ (which in general will depend on θ).