Statistical Data Analysis
Problem sheet 9
Due Monday, 9 December 2024

Exercise 1: Suppose the outcome of a measurement consists of a single value t modeled as following an exponential pdf

$$f(t|\tau) = \frac{1}{\tau} e^{-t/\tau} \,, \qquad (t \geq 0).$$

(a) [**3 marks**] Write down the log-likelihood function and find the Maximum-Likelihood estimator for the parameter τ.

(b) We observe a single value t and want to test hypothetical values of τ.

> **(i)** [**3 marks**] Take the critical region of the test to be $t > t_{\mathrm{cut}}$. Find the value of t_{cut} needed to have a test of size α.
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> **(ii)** [**2 marks**] For a given observed value t, find the corresponding p-value of a hypothesized value of τ, taking larger values of t to constitute increasing incompatibility with τ.
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> **(iii)** [**2 marks**] Suppose $t = 1\,\mathrm{s}$. Find the lower limit on τ at a confidence level of $\mathrm{CL} = 95\%$. Evaluate numerically.

(c) [**4 marks**] Consider the Bayesian approach to inference about τ. Using the second derivative of the log-likelihood, show that the Jeffreys prior for τ is

$$\pi(\tau) \propto \frac{1}{\tau} \qquad (\tau > 0).$$

(d) [**3 marks**] Using the Jeffreys prior, show that the posterior pdf is

$$p(\tau|t) = \frac{t}{\tau^2} e^{-t/\tau} \,.$$

(e) [**3 marks**] Find the mode of the posterior pdf above and comment on why it is less than the Maximum Likelihood estimator.