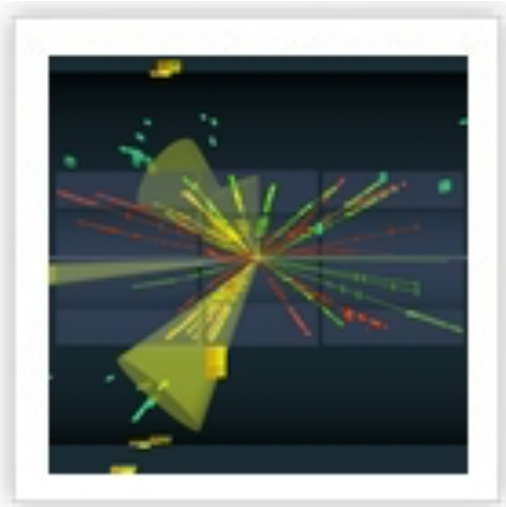


Statistical Methods for Particle Physics

Lecture 1: parameter estimation, statistical tests

<http://benasque.org/2019tae/>



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Outline

→ Lecture 1: Introduction and review of fundamentals

Probability, random variables, pdfs

Parameter estimation, maximum likelihood

Introduction to statistical tests

Lecture 2: More on statistical tests

Discovery, limits

Bayesian limits

Lecture 3: Framework for full analysis

Nuisance parameters and systematic uncertainties

Tests from profile likelihood ratio

Lecture 4: Further topics

More parameter estimation, Bayesian methods

Experimental sensitivity

Some statistics books, papers, etc.

G. Cowan, *Statistical Data Analysis*, Clarendon, Oxford, 1998

R.J. Barlow, *Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences*, Wiley, 1989

Ilya Narsky and Frank C. Porter, *Statistical Analysis Techniques in Particle Physics*, Wiley, 2014.

Luca Lista, *Statistical Methods for Data Analysis in Particle Physics*, Springer, 2017.

L. Lyons, *Statistics for Nuclear and Particle Physics*, CUP, 1986

F. James., *Statistical and Computational Methods in Experimental Physics*, 2nd ed., World Scientific, 2006

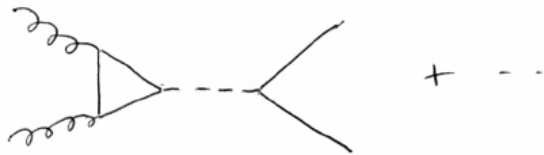
S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1998 (with program library on CD)

M. Tanabashi et al. (PDG), Phys. Rev. D 98, 030001 (2018); see also pdg.lbl.gov sections on probability, statistics, Monte Carlo

Theory ↔ Statistics ↔ Experiment

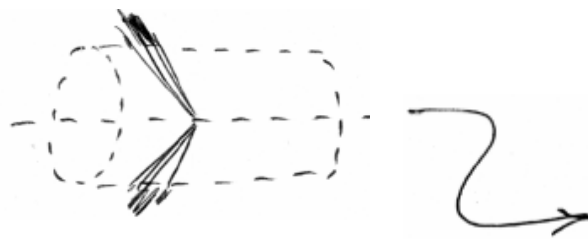
Theory (model, hypothesis):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi} \not{D} \Psi + \dots$$

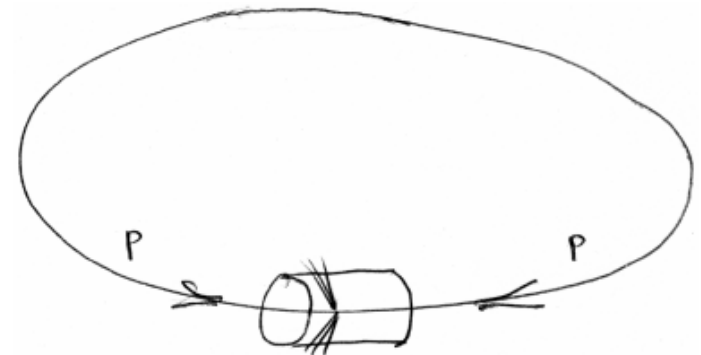


$$\sigma = \frac{G_F \alpha_s^2 m_H^2}{288 \sqrt{2}\pi} \times \text{wavy line}$$

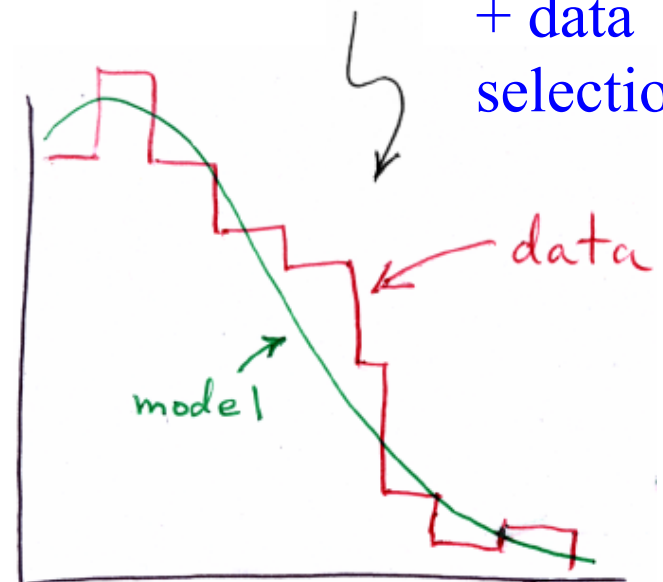
+ simulation
of detector
and cuts



Experiment:



+ data
selection



Data analysis in particle physics

Observe events (e.g., pp collisions) and for each, measure a set of characteristics:

particle momenta, number of muons, energy of jets,...

Compare observed distributions of these characteristics to predictions of theory. From this, we want to:

Estimate the free parameters of the theory: $m_H = 125.4$

Quantify the uncertainty in the estimates: $\pm 0.4 \text{ GeV}$

Assess how well a given theory stands in agreement with the observed data:

0^+ good, 2^+ bad

To do this we need a clear definition of **PROBABILITY**

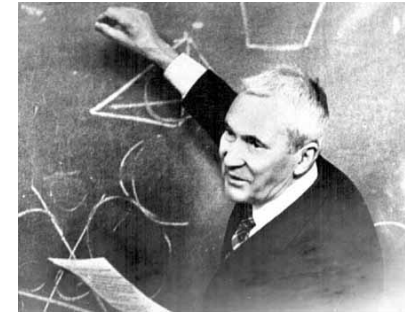
A definition of probability

Consider a set S with subsets A, B, \dots

For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$



**Kolmogorov
axioms (1933)**

Also define **conditional
probability of A given B :**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Subsets A, B **independent** if: $P(A \cap B) = P(A)P(B)$

If A, B independent, $P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$

Interpretation of probability

I. Relative frequency

A, B, \dots are outcomes of a repeatable experiment

$$P(A) = \lim_{n \rightarrow \infty} \frac{\text{times outcome is } A}{n}$$

cf. quantum mechanics, particle scattering, radioactive decay...

II. Subjective probability

A, B, \dots are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

- Both interpretations consistent with Kolmogorov axioms.
- In particle physics frequency interpretation often most useful, but subjective probability can provide more natural treatment of non-repeatable phenomena:

systematic uncertainties, probability that Higgs boson exists,...

Bayes' theorem

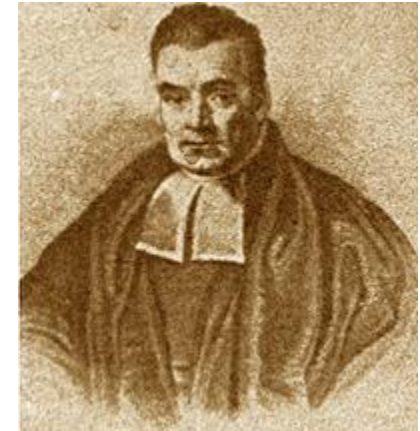
From the definition of conditional probability we have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

but $P(A \cap B) = P(B \cap A)$, so

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' theorem



First published (posthumously) by the Reverend Thomas Bayes (1702–1761)

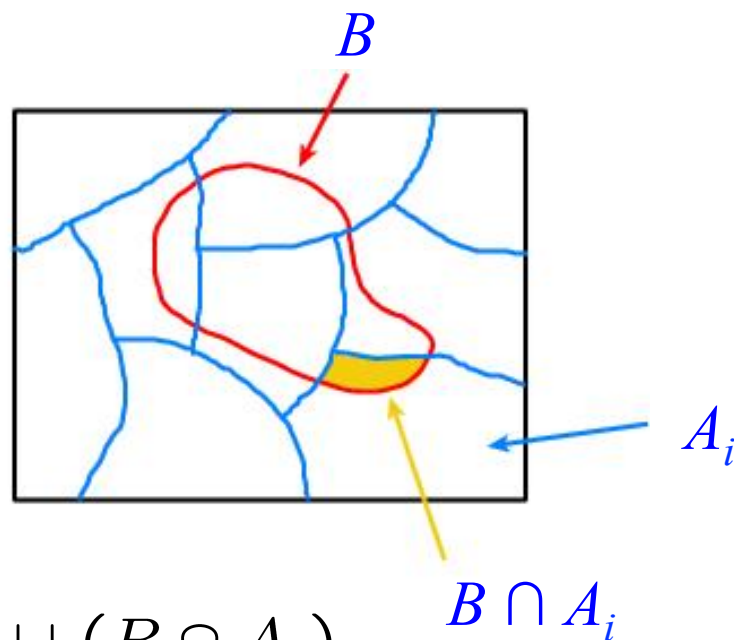
An essay towards solving a problem in the doctrine of chances, Philos. Trans. R. Soc. **53** (1763) 370; reprinted in Biometrika, **45** (1958) 293.

The law of total probability

Consider a subset B of the sample space S ,

divided into disjoint subsets A_i such that $\cup_i A_i = S$,

S →



$$\rightarrow B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i),$$

$$\rightarrow P(B) = P(\cup_i (B \cap A_i)) = \sum_i P(B \cap A_i)$$

$$\rightarrow P(B) = \sum_i P(B|A_i)P(A_i) \quad \text{law of total probability}$$

Bayes' theorem becomes

$$P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

Bayes' theorem example (cont.)

The probability to have the disease given a + result is

$$\begin{aligned} p(\text{D}|+) &= \frac{P(+|\text{D})P(\text{D})}{P(+|\text{D})P(\text{D}) + P(+|\text{no D})P(\text{no D})} \\ &= \frac{0.98 \times 0.001}{0.98 \times 0.001 + 0.03 \times 0.999} \\ &= 0.032 \quad \leftarrow \text{posterior probability} \end{aligned}$$

i.e. you're probably OK!

Your viewpoint: my degree of belief that I have the disease is 3.2%.

Your doctor's viewpoint: 3.2% of people like this have the disease.

Frequentist Statistics – general philosophy

In frequentist statistics, probabilities are associated only with the data, i.e., outcomes of repeatable observations (shorthand: \vec{x}).

Probability = limiting frequency

Probabilities such as

P (Higgs boson exists),

$P(0.117 < \alpha_s < 0.121)$,

etc. are either 0 or 1, but we don't know which.

The tools of frequentist statistics tell us what to expect, under the assumption of certain probabilities, about hypothetical repeated observations.

A hypothesis is preferred if the data are found in a region of high predicted probability (i.e., where an alternative hypothesis predicts lower probability).

Bayesian Statistics – general philosophy

In Bayesian statistics, use subjective probability for hypotheses:

probability of the data assuming hypothesis H (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H) dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

Bayes' theorem has an “if-then” character: **If** your prior probabilities were $\pi(H)$, **then** it says how these probabilities should change in the light of the data.

No general prescription for priors (subjective!)

The likelihood function

Suppose the entire result of an experiment (set of measurements) is a collection of numbers \mathbf{x} , and suppose the joint pdf for the data \mathbf{x} is a function that depends on a set of parameters θ :

$$P(\mathbf{x}|\theta)$$

Now evaluate this function with the data obtained and regard it as a function of the parameter(s). This is the **likelihood function**:

$$L(\theta) = P(\mathbf{x}|\theta)$$

(\mathbf{x} constant)

The likelihood function for i.i.d.*. data

* i.i.d. = independent and identically distributed

Consider n independent observations of x : x_1, \dots, x_n , where x follows $f(x; \theta)$. The joint pdf for the whole data sample is:

$$f(x_1, \dots, x_n; \theta) = \prod_{i=1}^n f(x_i; \theta)$$

In this case the likelihood function is

$$L(\vec{\theta}) = \prod_{i=1}^n f(x_i; \vec{\theta}) \quad (x_i \text{ constant})$$

Frequentist parameter estimation

Suppose we have a pdf characterized by one or more parameters:

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}$$

random variable parameter

Suppose we have a **sample** of observed values: $\vec{x} = (x_1, \dots, x_n)$

We want to find some function of the data to **estimate** the parameter(s):

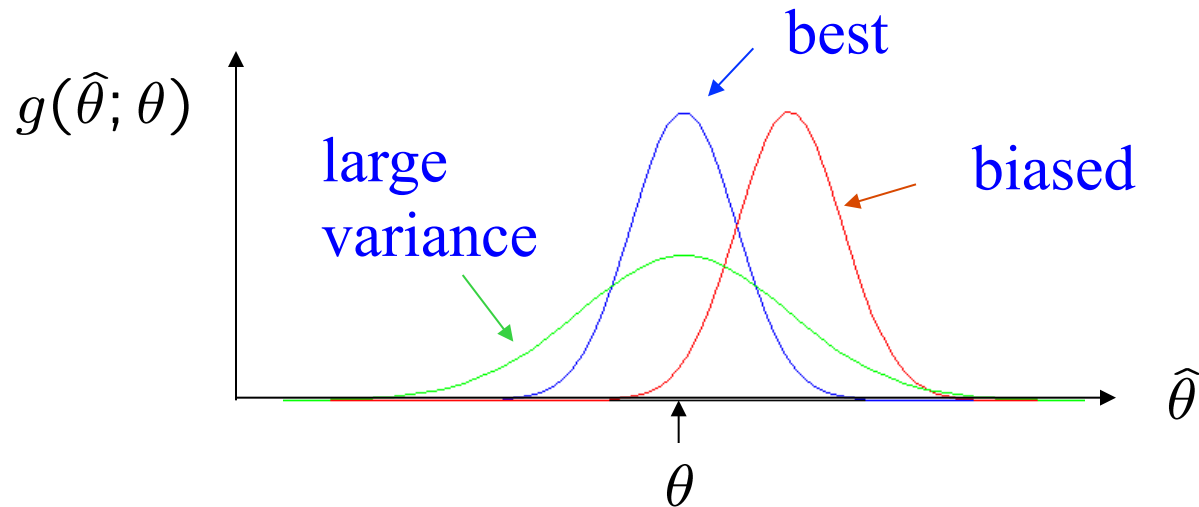
$$\hat{\theta}(\vec{x})$$

← estimator written with a hat

Sometimes we say ‘estimator’ for the function of x_1, \dots, x_n ;
‘estimate’ for the value of the estimator with a particular data set.

Properties of estimators

Estimators are functions of the data and thus characterized by a sampling distribution with a given (co)variance:

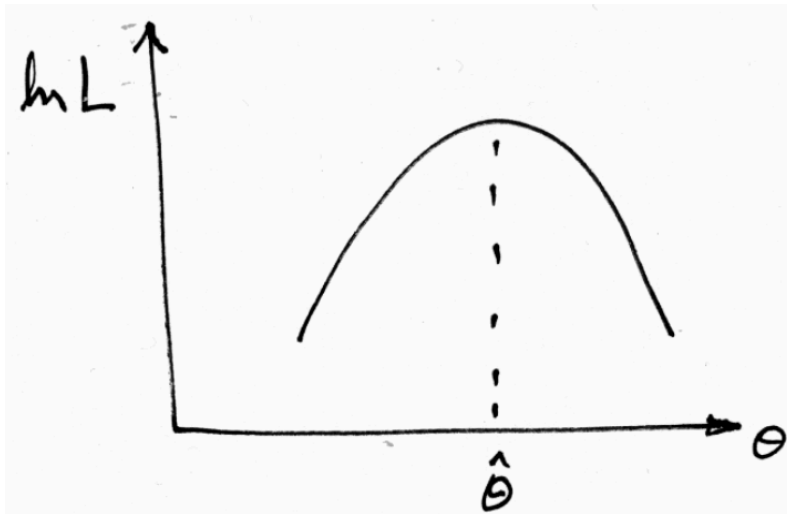


In general they may have a nonzero bias: $b = E[\hat{\theta}] - \theta$

Want small variance and small bias, but in general cannot optimize with respect to both; some trade-off necessary.

Maximum Likelihood (ML) estimators

The most important frequentist method for constructing estimators is to take the value of the parameter(s) that maximize the likelihood (or equivalently the log-likelihood):



$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} L(x|\theta)$$

In some cases we can find the ML estimator as a closed-form function of the data; more often it is found numerically.

ML example: parameter of exponential pdf

Consider exponential pdf, $f(t; \tau) = \frac{1}{\tau} e^{-t/\tau}$

and suppose we have i.i.d. data, t_1, \dots, t_n

The likelihood function is $L(\tau) = \prod_{i=1}^n \frac{1}{\tau} e^{-t_i/\tau}$

The value of τ for which $L(\tau)$ is maximum also gives the maximum value of its logarithm (the log-likelihood function):

$$\ln L(\tau) = \sum_{i=1}^n \ln f(t_i; \tau) = \sum_{i=1}^n \left(\ln \frac{1}{\tau} - \frac{t_i}{\tau} \right)$$

ML example: parameter of exponential pdf (2)

Find its maximum by setting $\frac{\partial \ln L(\tau)}{\partial \tau} = 0$,

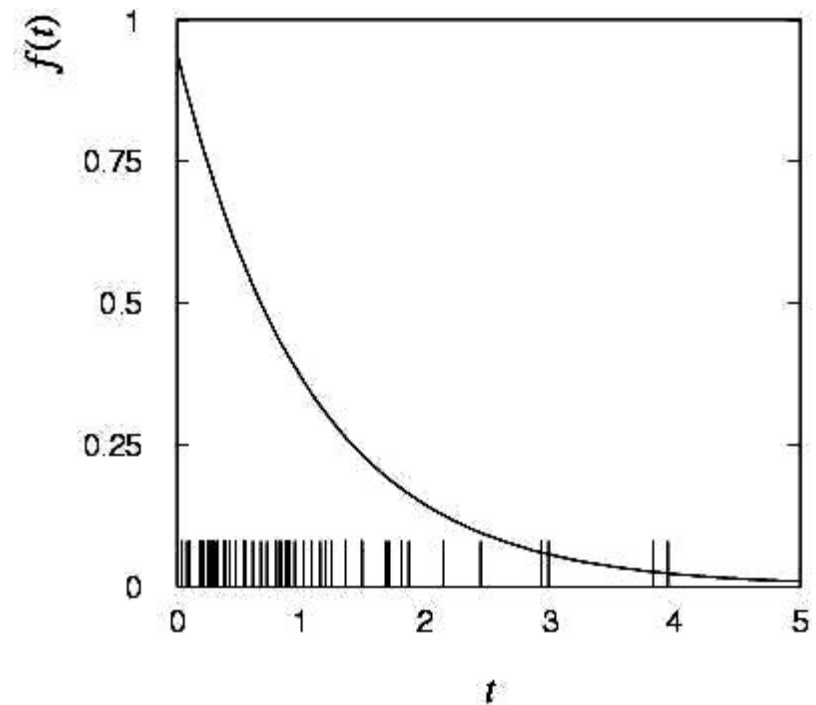
$$\rightarrow \hat{\tau} = \frac{1}{n} \sum_{i=1}^n t_i$$

Monte Carlo test:

generate 50 values
using $\tau = 1$:

We find the ML estimate:

$$\hat{\tau} = 1.062$$



ML example: parameter of exponential pdf (3)

For the exponential distribution one has for mean, variance:

$$E[t] = \int_0^{\infty} t \frac{1}{\tau} e^{-t/\tau} dt = \tau$$

$$V[t] = \int_0^{\infty} (t - \tau)^2 \frac{1}{\tau} e^{-t/\tau} dt = \tau^2$$

For the ML estimator $\hat{\tau} = \frac{1}{n} \sum_{i=1}^n t_i$ we therefore find

$$E[\hat{\tau}] = E \left[\frac{1}{n} \sum_{i=1}^n t_i \right] = \frac{1}{n} \sum_{i=1}^n E[t_i] = \tau \quad \longrightarrow \quad b = E[\hat{\tau}] - \tau = 0$$

$$V[\hat{\tau}] = V \left[\frac{1}{n} \sum_{i=1}^n t_i \right] = \frac{1}{n^2} \sum_{i=1}^n V[t_i] = \frac{\tau^2}{n} \quad \longrightarrow \quad \sigma_{\hat{\tau}} = \frac{\tau}{\sqrt{n}}$$

Variance of estimators: Monte Carlo method

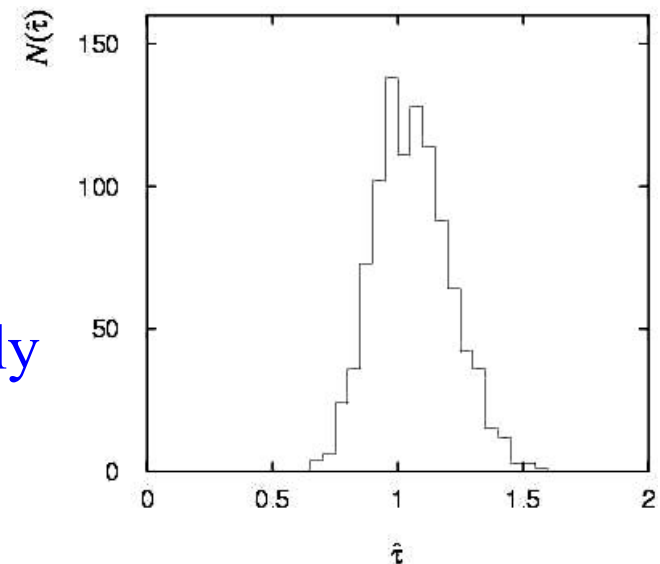
Having estimated our parameter we now need to report its ‘statistical error’, i.e., how widely distributed would estimates be if we were to repeat the entire measurement many times.

One way to do this would be to simulate the entire experiment many times with a Monte Carlo program (use ML estimate for MC).

For exponential example, from sample variance of estimates we find:

$$\hat{\sigma}_{\hat{\tau}} = 0.151$$

Note distribution of estimates is roughly Gaussian – (almost) always true for ML in large sample limit.



Variance of estimators from information inequality

The **information inequality** (RCF) sets a lower bound on the variance of any estimator (not only ML):

$$V[\hat{\theta}] \geq \left(1 + \frac{\partial b}{\partial \theta}\right)^2 \bigg/ E \left[-\frac{\partial^2 \ln L}{\partial \theta^2} \right]$$

← Minimum Variance Bound (MVB)
($b = E[\hat{\theta}] - \theta$)

Often the bias b is small, and equality either holds exactly or is a good approximation (e.g. large data sample limit). Then,

$$V[\hat{\theta}] \approx -1 \bigg/ E \left[\frac{\partial^2 \ln L}{\partial \theta^2} \right]$$

Estimate this using the 2nd derivative of $\ln L$ at its maximum:

$$\hat{V}[\hat{\theta}] = - \left(\frac{\partial^2 \ln L}{\partial \theta^2} \right)^{-1} \bigg|_{\theta=\hat{\theta}}$$

Variance of estimators: graphical method

Expand $\ln L(\theta)$ about its maximum:

$$\ln L(\theta) = \ln L(\hat{\theta}) + \left[\frac{\partial \ln L}{\partial \theta} \right]_{\theta=\hat{\theta}} (\theta - \hat{\theta}) + \frac{1}{2!} \left[\frac{\partial^2 \ln L}{\partial \theta^2} \right]_{\theta=\hat{\theta}} (\theta - \hat{\theta})^2 + \dots$$

First term is $\ln L_{\max}$, second term is zero, for third term use information inequality (assume equality):

$$\ln L(\theta) \approx \ln L_{\max} - \frac{(\theta - \hat{\theta})^2}{2\widehat{\sigma}_{\hat{\theta}}^2}$$

$$\text{i.e., } \ln L(\hat{\theta} \pm \hat{\sigma}_{\hat{\theta}}) \approx \ln L_{\max} - \frac{1}{2}$$

→ to get $\hat{\sigma}_{\hat{\theta}}$, change θ away from $\hat{\theta}$ until $\ln L$ decreases by 1/2.

Example of variance by graphical method

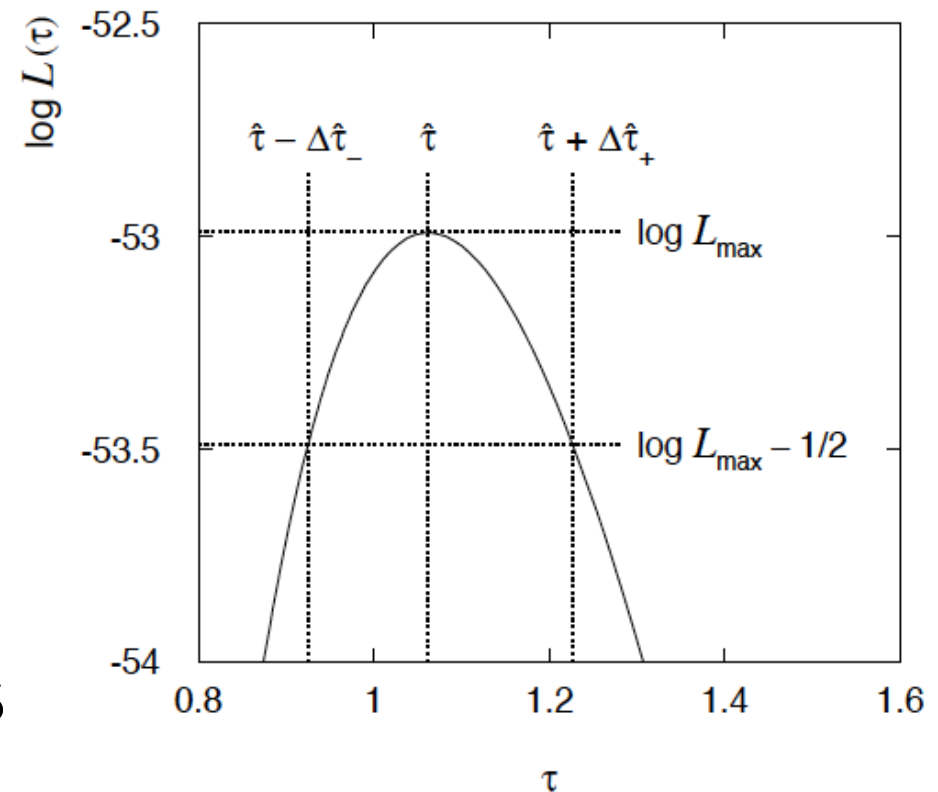
ML example with exponential:

$$\hat{\tau} = 1.062$$

$$\Delta \hat{\tau}_- = 0.137$$

$$\Delta \hat{\tau}_+ = 0.165$$

$$\hat{\sigma}_{\hat{\tau}} \approx \Delta \hat{\tau}_- \approx \Delta \hat{\tau}_+ \approx 0.15$$



Not quite parabolic $\ln L$ since finite sample size ($n = 50$).

Information inequality for N parameters

Suppose we have estimated N parameters $\vec{\theta} = (\theta_1, \dots, \theta_N)$.

The (inverse) minimum variance bound is given by the Fisher information matrix:

$$I_{ij} = E \left[-\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right] = -n \int f(x; \vec{\theta}) \frac{\partial^2 \ln f(x; \vec{\theta})}{\partial \theta_i \partial \theta_j} dx$$

The information inequality then states that $V - I^{-1}$ is a positive semi-definite matrix, where $V_{ij} = \text{cov}[\hat{\theta}_i, \hat{\theta}_j]$. Therefore

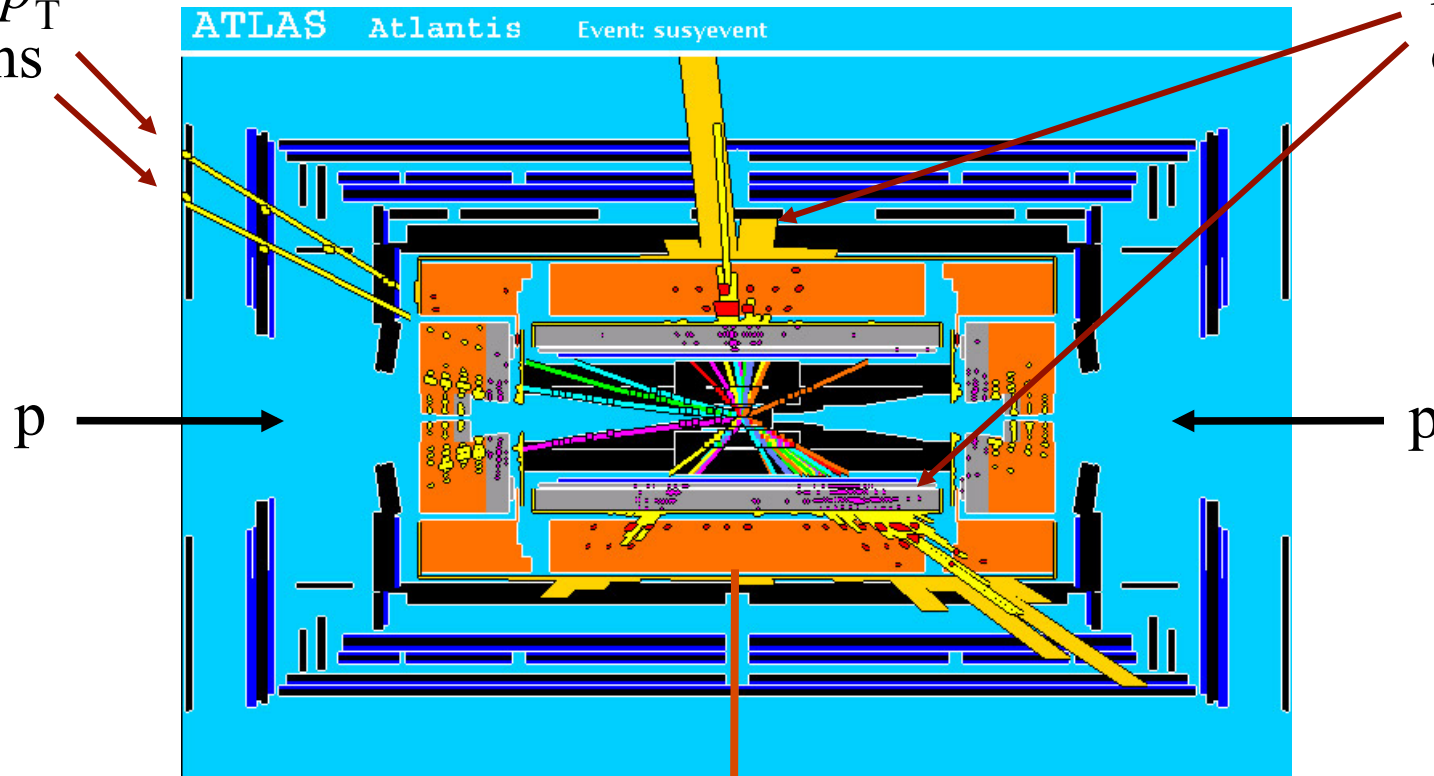
$$V[\hat{\theta}_i] \geq (I^{-1})_{ii}$$

Often use I^{-1} as an approximation for covariance matrix, estimate using e.g. matrix of 2nd derivatives at maximum of L .

Prelude to statistical tests: A simulated SUSY event

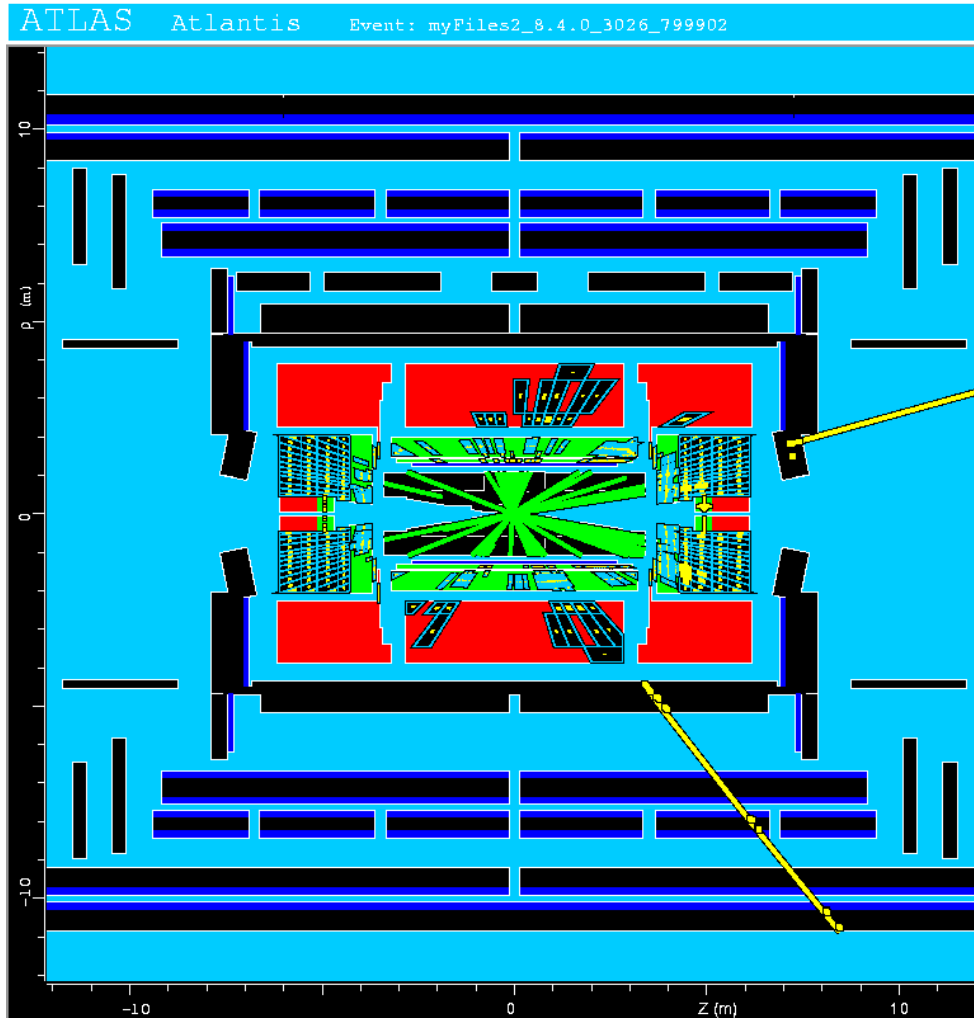
high p_T
muons

high p_T jets
of hadrons



missing transverse energy

Background events



This event from Standard Model $t\bar{t}$ production also has high p_T jets and muons, and some missing transverse energy.

→ can easily mimic a SUSY event.

Frequentist statistical tests

Suppose a measurement produces data \mathbf{x} ; consider a hypothesis H_0 we want to test and alternative H_1

H_0, H_1 specify probability for \mathbf{x} : $P(\mathbf{x}|H_0), P(\mathbf{x}|H_1)$

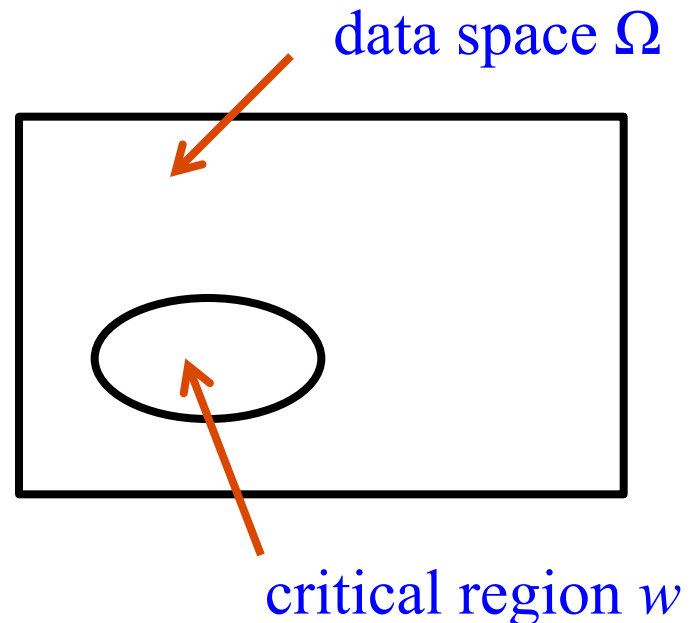
A test of H_0 is defined by specifying a **critical region** w of the data space such that there is no more than some (small) probability α , assuming H_0 is correct, to observe the data there, i.e.,

$$P(x \in w | H_0) \leq \alpha$$

Need inequality if data are discrete.

α is called the **size** or **significance level** of the test.

If x is observed in the critical region, reject H_0 .

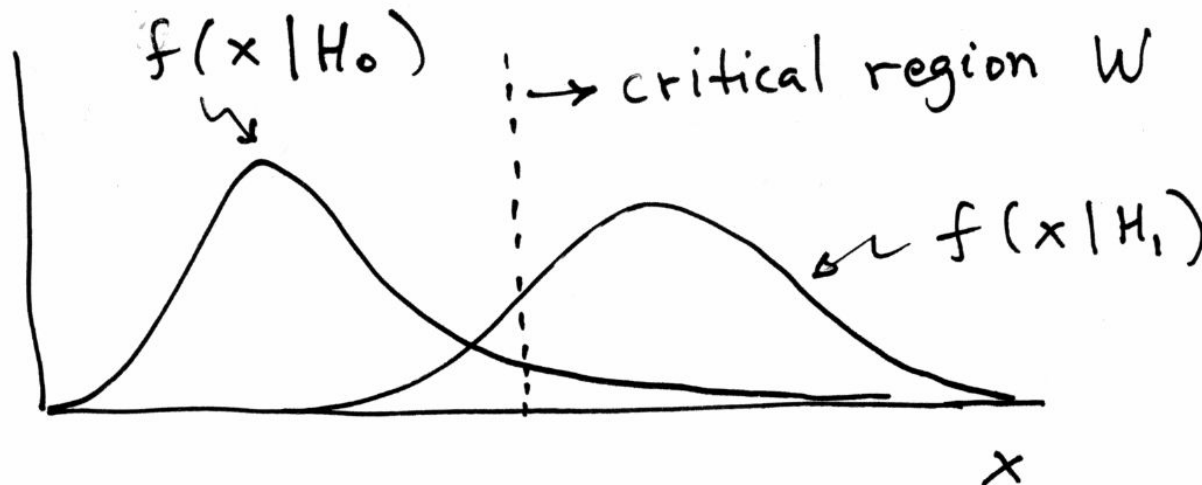


Definition of a test (2)

But in general there are an infinite number of possible critical regions that give the same significance level α .

So the choice of the critical region for a test of H_0 needs to take into account the alternative hypothesis H_1 .

Roughly speaking, place the critical region where there is a low probability to be found if H_0 is true, but high if H_1 is true:



Efficiencies, purity

Let $H_0 = b$ (event is background, $H_1 = s$ (event is signal).

For each event test b . If b rejected, “accept” as candidate signal.

background efficiency = $\varepsilon_b = P(\mathbf{x} \in W | b) = \alpha$

signal efficiency = $\varepsilon_s = \text{power} = P(\mathbf{x} \in W | s) = 1 - \beta$

To find purity of candidate signal sample, use Bayes’ theorem:

Here W is signal region

ε_s prior probability

$$P(s | \mathbf{x} \in W) = \frac{P(\mathbf{x} \in W | s)P(s)}{P(\mathbf{x} \in W | s)P(s) + P(\mathbf{x} \in W | b)P(b)}$$

posterior probability = signal purity ε_b

Physics context of a statistical test

Event Selection: data = individual event; goal is to classify

Example: separation of different particle types (electron vs muon) or known event types (ttbar vs QCD multijet).

E.g. test H_0 : event is background vs. H_1 : event is signal.

Use selected events for further study.

Search for New Physics: data = a sample of events. Test null hypothesis

H_0 : all events correspond to Standard Model (background only),

against the alternative

H_1 : events include a type whose existence is not yet established (signal plus background)

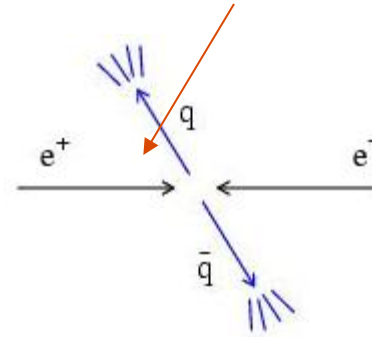
Many subtle issues here, mainly related to the high standard of proof required to establish presence of a new phenomenon. The optimal statistical test for a search is closely related to that used for event selection.

Extra slides

Example of ML with 2 parameters

Consider a scattering angle distribution with $x = \cos \theta$,

$$f(x; \alpha, \beta) = \frac{1 + \alpha x + \beta x^2}{2 + 2\beta/3}$$



or if $x_{\min} < x < x_{\max}$, need always to normalize so that

$$\int_{x_{\min}}^{x_{\max}} f(x; \alpha, \beta) dx = 1 .$$

Example: $\alpha = 0.5$, $\beta = 0.5$, $x_{\min} = -0.95$, $x_{\max} = 0.95$,
generate $n = 2000$ events with Monte Carlo.

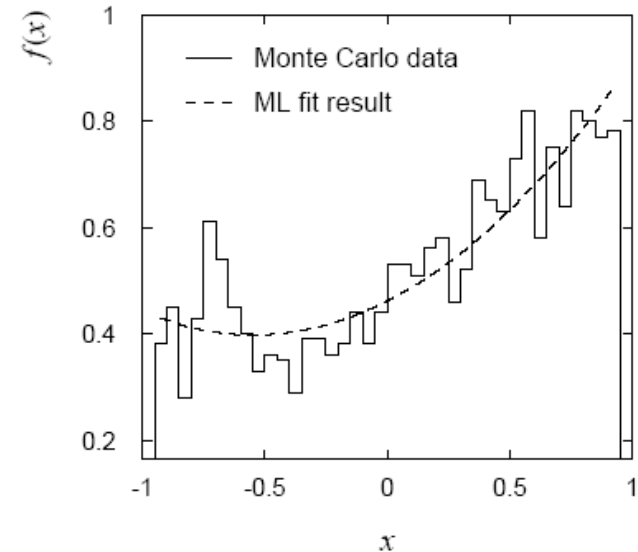
Example of ML with 2 parameters: fit result

Finding maximum of $\ln L(\alpha, \beta)$ numerically (**MINUIT**) gives

$$\hat{\alpha} = 0.508$$

$$\hat{\beta} = 0.47$$

N.B. No binning of data for fit, but can compare to histogram for goodness-of-fit (e.g. ‘visual’ or χ^2).



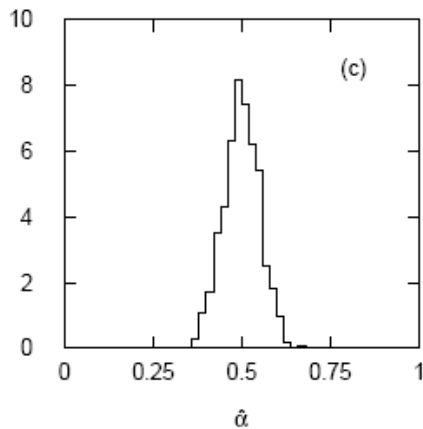
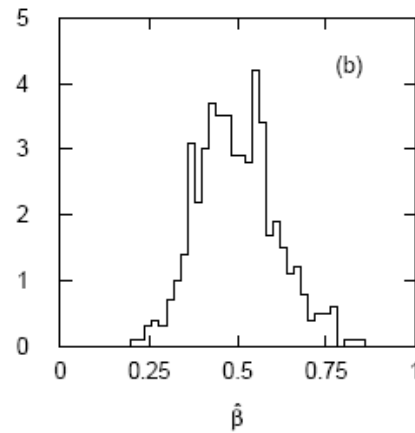
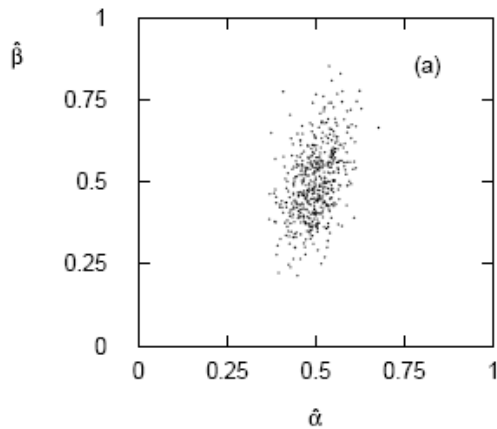
(Co)variances from $(\widehat{V}^{-1})_{ij} = -\left. \frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right|_{\vec{\theta}=\vec{\hat{\theta}}}$ (**MINUIT routine HESSE**)

$$\hat{\sigma}_{\hat{\alpha}} = 0.052 \quad \text{cov}[\hat{\alpha}, \hat{\beta}] = 0.0026$$

$$\hat{\sigma}_{\hat{\beta}} = 0.11 \quad r = 0.46$$

Two-parameter fit: MC study

Repeat ML fit with 500 experiments, all with $n = 2000$ events:



$$\overline{\hat{\alpha}} = 0.499$$

$$s_{\hat{\alpha}} = 0.051$$

$$\overline{\hat{\beta}} = 0.498$$

$$s_{\hat{\beta}} = 0.111$$

$$\text{cov}[\hat{\alpha}, \hat{\beta}] = 0.0024$$

$$r = 0.42$$

Estimates average to \sim true values;
(Co)variances close to previous estimates;
marginal pdfs approximately Gaussian.

The $\ln L_{\max} - 1/2$ contour

For large n , $\ln L$ takes on quadratic form near maximum:

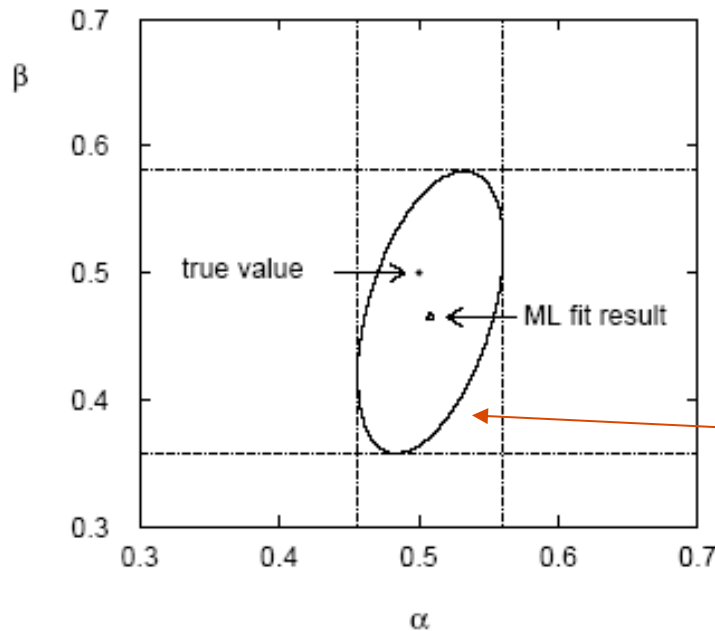
$$\ln L(\alpha, \beta) \approx \ln L_{\max}$$

$$-\frac{1}{2(1-\rho^2)} \left[\left(\frac{\alpha - \hat{\alpha}}{\sigma_{\hat{\alpha}}} \right)^2 + \left(\frac{\beta - \hat{\beta}}{\sigma_{\hat{\beta}}} \right)^2 - 2\rho \left(\frac{\alpha - \hat{\alpha}}{\sigma_{\hat{\alpha}}} \right) \left(\frac{\beta - \hat{\beta}}{\sigma_{\hat{\beta}}} \right) \right]$$

The contour $\ln L(\alpha, \beta) = \ln L_{\max} - 1/2$ is an ellipse:

$$\frac{1}{(1-\rho^2)} \left[\left(\frac{\alpha - \hat{\alpha}}{\sigma_{\hat{\alpha}}} \right)^2 + \left(\frac{\beta - \hat{\beta}}{\sigma_{\hat{\beta}}} \right)^2 - 2\rho \left(\frac{\alpha - \hat{\alpha}}{\sigma_{\hat{\alpha}}} \right) \left(\frac{\beta - \hat{\beta}}{\sigma_{\hat{\beta}}} \right) \right] = 1$$

(Co)variances from $\ln L$ contour



The α, β plane for the first MC data set

$$\ln L(\alpha, \beta) = \ln L_{\max} - 1/2$$

→ Tangent lines to contours give standard deviations.

→ Angle of ellipse ϕ related to correlation: $\tan 2\phi = \frac{2\rho\sigma_{\hat{\alpha}}\sigma_{\hat{\beta}}}{\sigma_{\hat{\alpha}}^2 - \sigma_{\hat{\beta}}^2}$

Correlations between estimators result in an increase in their standard deviations (statistical errors).