

Lectures on Statistical Data Analysis



London Postgraduate Lectures on Particle Physics;
University of London MSci course PH4515



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Course web page:

`www.pp.rhul.ac.uk/~cowan/stat_course.html`

Statistical Data Analysis: Outline by Lecture

- 1 Probability, Bayes' theorem
- 2 Random variables and probability densities
- 3 Expectation values, error propagation
- 4 Catalogue of pdfs
- 5 The Monte Carlo method
- 6 Statistical tests: general concepts
- 7 Test statistics, multivariate methods
- 8 Goodness-of-fit tests
- 9 Parameter estimation, maximum likelihood
- 10 More maximum likelihood
- 11 Method of least squares
- 12 Interval estimation, setting limits
- 13 Nuisance parameters, systematic uncertainties
- 14 Examples of Bayesian approach

Some statistics books, papers, etc.

G. Cowan, *Statistical Data Analysis*, Clarendon, Oxford, 1998

see also www.pp.rhul.ac.uk/~cowan/sda

R.J. Barlow, *Statistics, A Guide to the Use of Statistical in the Physical Sciences*, Wiley, 1989

see also hepwww.ph.man.ac.uk/~roger/book.html

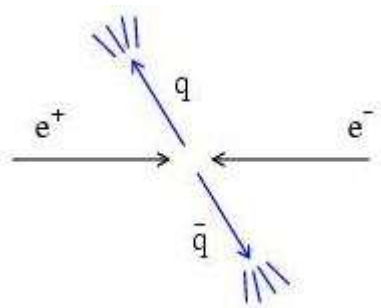
L. Lyons, *Statistics for Nuclear and Particle Physics*, CUP, 1986

F. James., *Statistical and Computational Methods in Experimental Physics*, 2nd ed., World Scientific, 2006

S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1998 (with program library on CD)

W.M. Yao et al. (Particle Data Group), *Review of Particle Physics*, Journal of Physics G 33 (2006) 1; see also pdg.lbl.gov sections on probability statistics, Monte Carlo

Data analysis in particle physics



Observe events of a certain type

Measure characteristics of each event (particle momenta, number of muons, energy of jets,...)

Theories (e.g. SM) predict distributions of these properties up to free parameters, e.g., α , G_F , M_Z , α_s , m_H , ...

Some tasks of data analysis:

Estimate (measure) the parameters;

Quantify the uncertainty of the parameter estimates;

Test the extent to which the predictions of a theory are in agreement with the data.

Dealing with uncertainty

In particle physics there are various elements of uncertainty:

theory is not deterministic

quantum mechanics

random measurement errors

present even without quantum effects

things we could know in principle but don't

e.g. from limitations of cost, time, ...



We can quantify the uncertainty using **PROBABILITY**

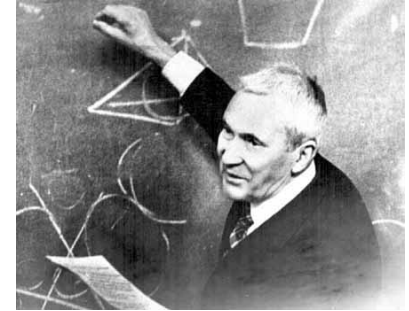
A definition of probability

Consider a set S with subsets A, B, \dots

For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$



Kolmogorov
axioms (1933)

From these axioms we can derive further properties, e.g.

$$P(\bar{A}) = 1 - P(A)$$

$$P(A \cup \bar{A}) = 1$$

$$P(\emptyset) = 0$$

if $A \subset B$, then $P(A) \leq P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability, independence

Also define conditional probability of A given B (with $P(B) \neq 0$):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

E.g. rolling dice: $P(n < 3 | n \text{ even}) = \frac{P((n < 3) \cap n \text{ even})}{P(\text{even})} = \frac{1/6}{3/6} = \frac{1}{3}$

Subsets A, B independent if: $P(A \cap B) = P(A)P(B)$

If A, B independent, $P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$

N.B. do not confuse with disjoint subsets, i.e., $A \cap B = \emptyset$

Interpretation of probability

I. Relative frequency

A, B, \dots are outcomes of a repeatable experiment

$$P(A) = \lim_{n \rightarrow \infty} \frac{\text{times outcome is } A}{n}$$

cf. quantum mechanics, particle scattering, radioactive decay...

II. Subjective probability

A, B, \dots are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

- Both interpretations consistent with Kolmogorov axioms.
- In particle physics frequency interpretation often most useful, but subjective probability can provide more natural treatment of non-repeatable phenomena:

systematic uncertainties, probability that Higgs boson exists,...

Bayes' theorem

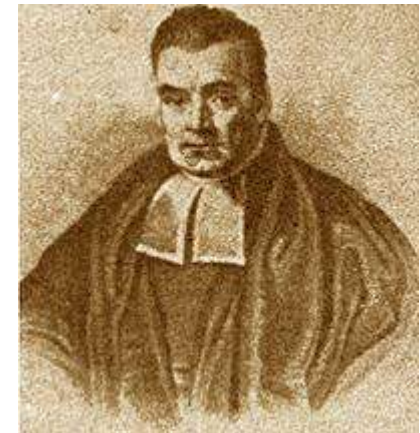
From the definition of conditional probability we have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

but $P(A \cap B) = P(B \cap A)$, so

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' theorem



First published (posthumously) by the Reverend Thomas Bayes (1702–1761)

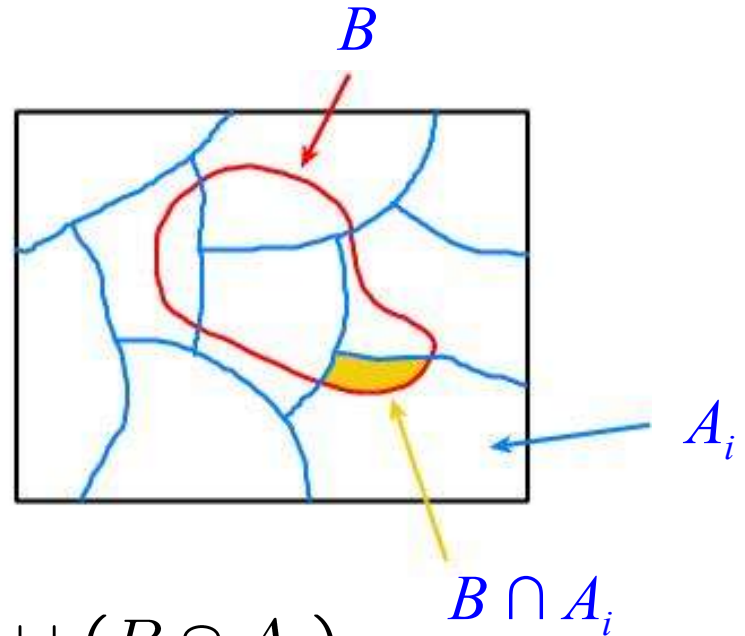
An essay towards solving a problem in the doctrine of chances, Philos. Trans. R. Soc. **53** (1763) 370; reprinted in Biometrika, **45** (1958) 293.

The law of total probability

Consider a subset B of the sample space S ,

divided into disjoint subsets A_i such that $\cup_i A_i = S$,

S



$$\rightarrow B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i),$$

$$\rightarrow P(B) = P(\cup_i (B \cap A_i)) = \sum_i P(B \cap A_i)$$

$$\rightarrow P(B) = \sum_i P(B|A_i)P(A_i) \quad \text{law of total probability}$$

Bayes' theorem becomes

$$P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

An example using Bayes' theorem

Suppose the probability (for anyone) to have AIDS is:

$$P(\text{AIDS}) = 0.001$$
$$P(\text{no AIDS}) = 0.999$$

← prior probabilities, i.e.,
before any test carried out

Consider an AIDS test: result is + or –

$$P(+|\text{AIDS}) = 0.98$$
$$P(-|\text{AIDS}) = 0.02$$

← probabilities to (in)correctly
identify an infected person

$$P(+|\text{no AIDS}) = 0.03$$
$$P(-|\text{no AIDS}) = 0.97$$

← probabilities to (in)correctly
identify an uninfected person

Suppose your result is +. How worried should you be?

Bayes' theorem example (cont.)

The probability to have AIDS given a + result is

$$\begin{aligned}P(\text{AIDS}|+) &= \frac{P(+|\text{AIDS})P(\text{AIDS})}{P(+|\text{AIDS})P(\text{AIDS}) + P(+|\text{no AIDS})P(\text{no AIDS})} \\ &= \frac{0.98 \times 0.001}{0.98 \times 0.001 + 0.03 \times 0.999} \\ &= 0.032 \quad \leftarrow \text{posterior probability}\end{aligned}$$

i.e. you're probably OK!

Your viewpoint: my degree of belief that I have AIDS is 3.2%

Your doctor's viewpoint: 3.2% of people like this will have AIDS

Frequentist Statistics – general philosophy

In frequentist statistics, probabilities are associated only with the data, i.e., outcomes of repeatable observations (shorthand: \vec{x}).

Probability = limiting frequency

Probabilities such as

P (Higgs boson exists),

$P(0.117 < \alpha_s < 0.121)$,

etc. are either 0 or 1, but we don't know which.

The tools of frequentist statistics tell us what to expect, under the assumption of certain probabilities, about hypothetical repeated observations.

The preferred theories (models, hypotheses, ...) are those for which our observations would be considered 'usual'.

Bayesian Statistics – general philosophy

In Bayesian statistics, use subjective probability for hypotheses:

probability of the data assuming hypothesis H (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H) dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

Bayes' theorem has an “if-then” character: **If** your prior probabilities were $\pi(H)$, **then** it says how these probabilities should change in the light of the data.

No general prescription for priors (subjective!)

Wrapping up lecture 1

Up to now we've talked some abstract properties of **probability**:

definition and interpretation,
Bayes' theorem, ...

Next we will look at *random variables* (numerical labels for the outcome of an experiment) and we will describe them using *probability density functions*.