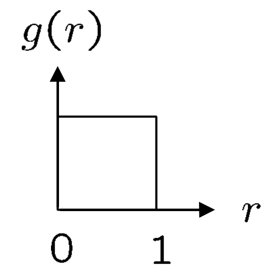
Statistical Data Analysis: Lecture 5

The Monte Carlo method

What it is: a numerical technique for calculating probabilities and related quantities using sequences of random numbers.

The usual steps:



(1) Generate sequence $r_1, r_2, \ldots, r_m$ uniform in [0, 1].

(2) Use this to produce another sequence $x_1, x_2, \ldots, x_n$ distributed according to some pdf $f(x)$ in which we're interested (x can be a vector).

(3) Use the x values to estimate some property of $f(x)$, e.g., fraction of x values with $a < x < b$ gives $\int_a^b f(x)\,dx$.

→ MC calculation = integration (at least formally)

MC generated values = 'simulated data'

→ use for testing statistical procedures

Random number generators

Goal: generate uniformly distributed values in [0, 1].

Toss coin for e.g. 32 bit number... (too tiring).

→ 'random number generator'

= computer algorithm to generate $r_1, r_2, \ldots, r_n$.

Example: multiplicative linear congruential generator (MLCG)

$n_{i+1} = (a\, n_i) \bmod m$, where

n_i = integer

a = multiplier

m = modulus

n_0 = seed (initial value)

N.B. mod = modulus (remainder), e.g. 27 mod 5 = 2.

This rule produces a sequence of numbers $n_0, n_1, \ldots$

Random number generators (2)

The sequence is (unfortunately) periodic!

Example (see Brandt Ch 4): $a = 3$, $m = 7$, $n_0 = 1$

$$n_1 = (3 \cdot 1)\text{mod}7 = 3$$

$$n_2 = (3 \cdot 3)\text{mod}7 = 2$$

$$n_3 = (3 \cdot 2)\text{mod}7 = 6$$

$$n_4 = (3 \cdot 6)\text{mod}7 = 4$$

$$n_5 = (3 \cdot 4)\text{mod}7 = 5$$

$$n_6 = (3 \cdot 5)\text{mod}7 = 1$$

← sequence repeats

Choose a, m to obtain long period (maximum = m – 1); m usually close to the largest integer that can represented in the computer.

Only use a subset of a single period of the sequence.

Random number generators (3)

$r_i = n_i/m$ are in [0, 1] but are they 'random'?

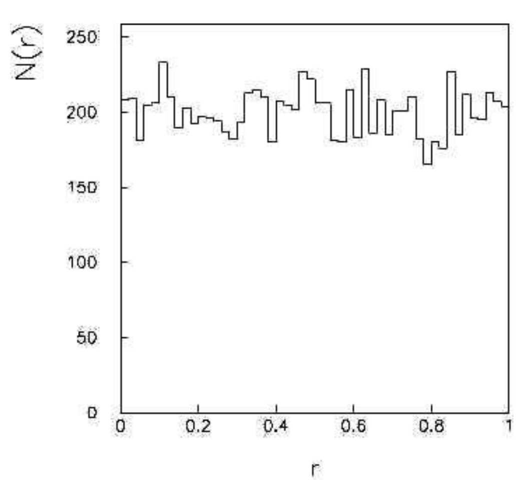
Choose a, m so that the r_i pass various tests of randomness:

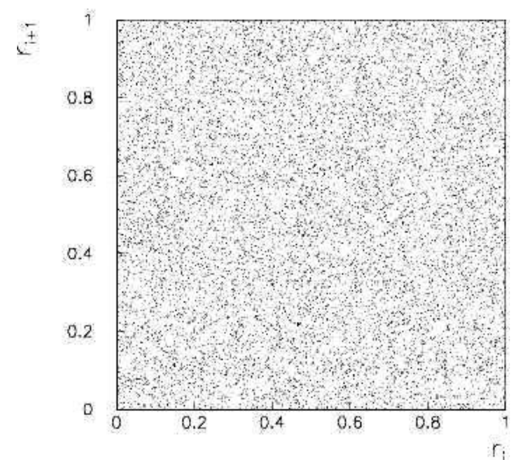
uniform distribution in [0, 1],

all values independent (no correlations between pairs),

e.g. L'Ecuyer, Commun. ACM **31** (1988) 742 suggests

$a = 40692$
$m = 2147483399$



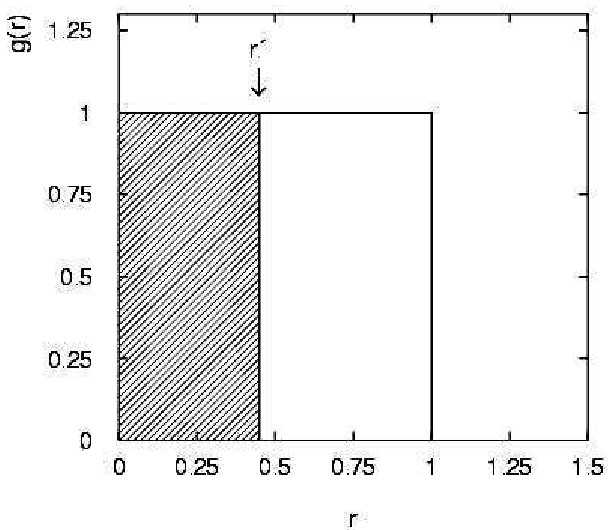


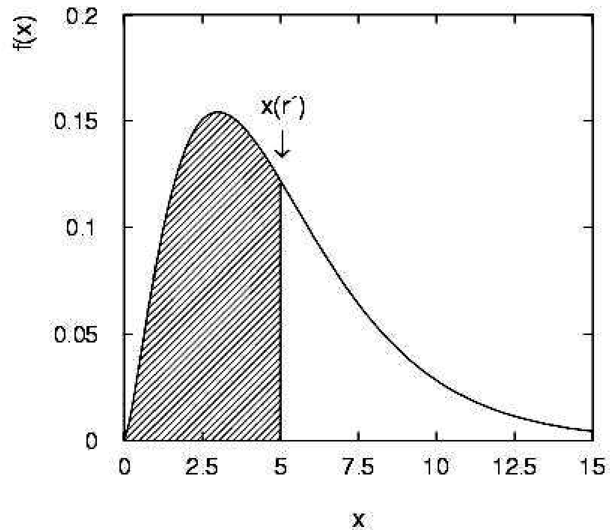
Far better algorithms available, e.g. `TRandom3`, period $\approx 10^{6000}$.

See F. James, Comp. Phys. Comm. 60 (1990) 111; Brandt Ch. 4

The transformation method

Given $r_1, r_2, \ldots, r_n$ uniform in [0, 1], find $x_1, x_2, \ldots, x_n$ that follow $f(x)$ by finding a suitable transformation $x(r)$.





Require: $P(r \leq r') = P(x \leq x(r'))$

i.e. $$\int_{-\infty}^{r'} g(r)\, dr = r' = \int_{-\infty}^{x(r')} f(x')\, dx' = F(x(r'))$$

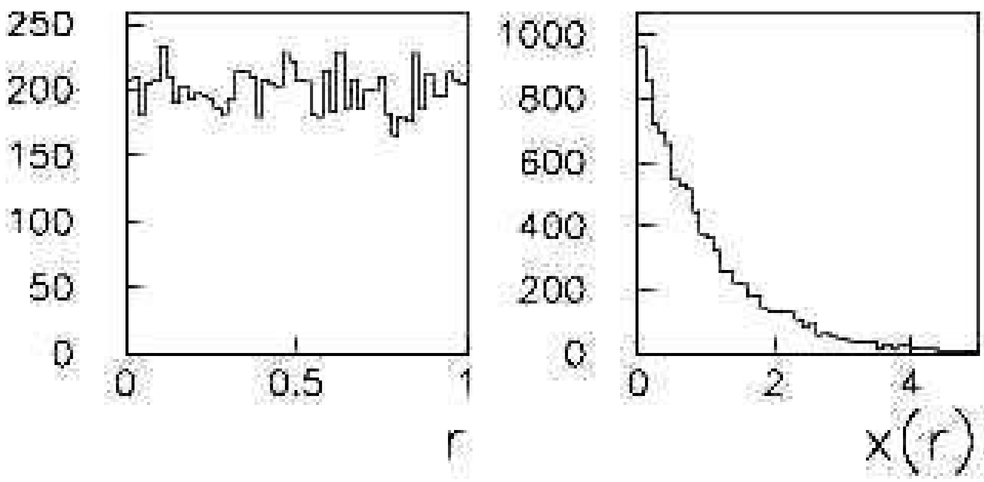
That is, set $F(x) = r$ and solve for $x(r)$.

Example of the transformation method

Exponential pdf: $f(x;\xi) = \frac{1}{\xi} e^{-x/\xi} \quad (x \geq 0)$

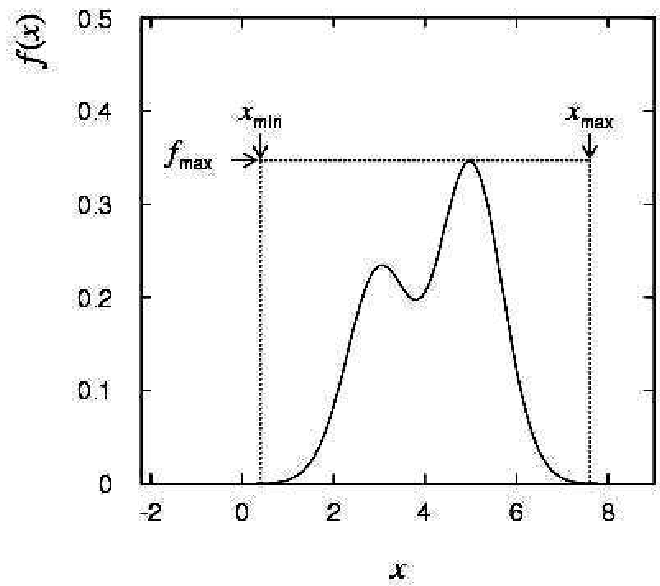
Set $\int_0^x \frac{1}{\xi} e^{-x'/\xi} \, dx' = r$ and solve for $x(r)$.

$\rightarrow \quad x(r) = -\xi \ln(1-r)$ $\quad$ ($x(r) = -\xi \ln r$ works too.)



The acceptance-rejection method

Enclose the pdf in a box:



(1) Generate a random number x, uniform in $[x_{min}, x_{max}]$, i.e.

$$x = x_{\mathsf{min}} + r_1(x_{\mathsf{max}} - x_{\mathsf{min}}) \; , \; r_1 \text{ is uniform in } [0,1].$$

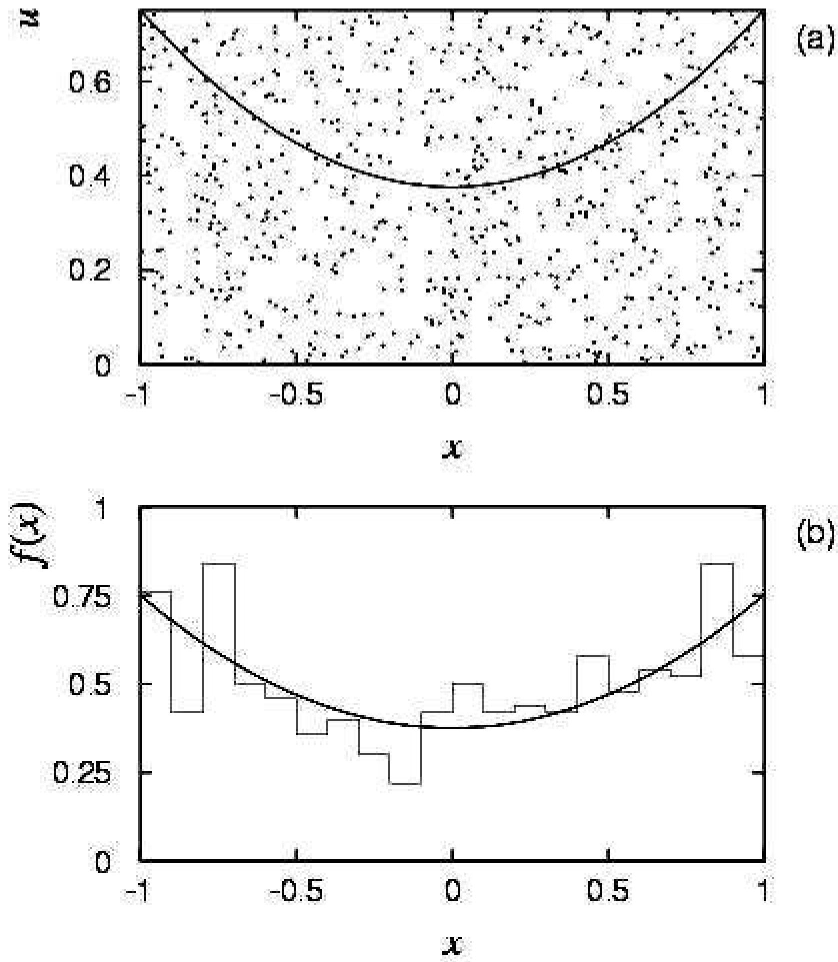
(2) Generate a 2nd independent random number u uniformly distributed between 0 and f_{max}, i.e. $u = r_2 f_{\mathsf{max}}$.

(3) If $u < f(x)$, then accept x. If not, reject x and repeat.

Example with acceptance-rejection method

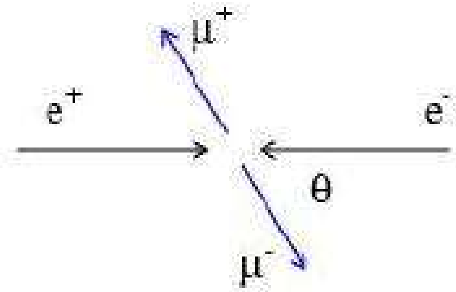
$$f(x) = \frac{3}{8}(1 + x^2)$$

$$(-1 \le x \le 1)$$



If dot below curve, use x value in histogram.

Monte Carlo event generators



Simple example: $e^+e^- \to \mu^+\mu^-$

Generate $\cos\theta$ and ϕ:

$$f(\cos\theta; A_{\mathrm{FB}}) \propto (1 + \frac{8}{3}A_{\mathrm{FB}}\cos\theta + \cos^2\theta) \, ,$$

$$g(\phi) = \frac{1}{2\pi} \quad (0 \le \phi \le 2\pi)$$

Less simple: ‘event generators’ for a variety of reactions:

$e^+e^- \to \mu^+\mu^-$, hadrons, ...

pp $\to$ hadrons, D-Y, SUSY,...

e.g. PYTHIA, HERWIG, ISAJET...

Output = ‘events’, i.e., for each event we get a list of generated particles and their momentum vectors, types, etc.

Monte Carlo detector simulation

Takes as input the particle list and momenta from generator.

Simulates detector response:

multiple Coulomb scattering (generate scattering angle),
particle decays (generate lifetime),
ionization energy loss (generate Δ),
electromagnetic, hadronic showers,
production of signals, electronics response, ...

Output = simulated raw data → input to reconstruction software:

track finding, fitting, etc.

Predict what you should see at 'detector level' given a certain hypothesis for 'generator level'. Compare with the real data.

Estimate 'efficiencies' = #events found / # events generated.

Programming package: `GEANT`

Wrapping up lecture 5

We've now seen the Monte Carlo method:

calculations based on sequences of random numbers,
used to simulate particle collisions, detector response.

So far, we've mainly been talking about probability.

But suppose now we are faced with experimental data.
We want to infer something about the (probabilistic) processes that produced the data.

This is statistics, the main subject of the following lectures.