

Statistical Methods for Particle Physics  
Solutions to problem on exponential fit

**1(a) [4 marks]** The exponentially distributed time measurements,  $t_1, \dots, t_n$ , and the Gaussian distributed calibration measurement  $y$  are all independent, so the likelihood is simply the product of the corresponding pdfs:

$$L(\tau, \lambda) = \prod_{i=1}^n \frac{1}{\tau + \lambda} e^{-t_i/(\tau+\lambda)} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\lambda)^2/2\sigma^2} .$$

The log-likelihood is therefore

$$\ln L(\tau, \lambda) = -n \ln(\tau + \lambda) - \frac{1}{\tau + \lambda} \sum_{i=1}^n t_i - \frac{(y - \lambda)^2}{2\sigma^2} + C ,$$

where  $C$  represents terms that do not depend on the parameters and therefore can be dropped. Differentiating  $\ln L$  with respect to the parameters gives

$$\begin{aligned} \frac{\partial \ln L}{\partial \tau} &= -\frac{n}{\tau + \lambda} + \frac{\sum_{i=1}^n t_i}{(\tau + \lambda)^2} \\ \frac{\partial \ln L}{\partial \lambda} &= -\frac{n}{\tau + \lambda} + \frac{\sum_{i=1}^n t_i}{(\tau + \lambda)^2} + \frac{y - \lambda}{\sigma^2} . \end{aligned}$$

Setting the derivatives to zero and solving for  $\tau$  and  $\lambda$  gives the ML estimators,

$$\begin{aligned} \hat{\tau} &= \frac{1}{n} \sum_{i=1}^n t_i - y \\ \hat{\lambda} &= y . \end{aligned}$$

**1(b) [4 marks]** The variances of  $\hat{\lambda}$  and  $\hat{\tau}$  and their covariance are

$$\begin{aligned} V[\hat{\lambda}] &= V[y] = \sigma^2 , \\ V[\hat{\tau}] &= V\left[\frac{1}{n} \sum_{i=1}^n t_i - y\right] = \frac{1}{n^2} \sum_{i=1}^n V[t_i] + V[y] = \frac{(\tau + \lambda)^2}{n} + \sigma^2 \\ \text{cov}[\hat{\tau}, \hat{\lambda}] &= \text{cov}\left[\frac{1}{n} \sum_{i=1}^n t_i - y, y\right] = -V[y] = -\sigma^2 , \end{aligned}$$

For the covariance we used the fact that  $t_i$  and  $y$  are independent and thus have zero covariance.

**1(c) [4 marks]** The standard deviations of  $\hat{\tau}$  and  $\hat{\lambda}$  can be determined from the contour of  $\ln L(\tau, \lambda) = \ln L_{\max} - 1/2$ , as shown in Fig. 1. The standard can be approximated by the distance from the maximum of  $\ln L$  to the tangent line to the contour (in either direction).

Figure 1: Illustration of the method to find  $\sigma_{\hat{\tau}}$  and  $\sigma_{\hat{\lambda}}$  from the contour of  $\ln L(\tau, \lambda) = \ln L_{\max} - 1/2$  (see text).

If  $\lambda$  were to be known exactly, then the standard deviation of  $\hat{\tau}$  would be less. This can be seen from Fig. 1, for example, since the distance one need to move  $\tau$  away from the maximum of  $\ln L$  to get to  $\ln L_{\max} - 1/2$  would be less if  $\lambda$  were to be fixed at  $\hat{\lambda}$ .

**1(d) [4 marks]** The second derivatives of  $\ln L$  are

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \tau^2} &= \frac{n}{(\tau + \lambda)^2} - \frac{2 \sum_{i=1}^n t_i}{(\tau + \lambda)^3}, \\ \frac{\partial^2 \ln L}{\partial \lambda^2} &= \frac{n}{(\tau + \lambda)^2} - \frac{2 \sum_{i=1}^n t_i}{(\tau + \lambda)^3} - \frac{1}{\sigma^2}, \\ \frac{\partial^2 \ln L}{\partial \tau \partial \lambda} &= \frac{n}{(\tau + \lambda)^2} - \frac{2 \sum_{i=1}^n t_i}{(\tau + \lambda)^3}.\end{aligned}$$

Using  $E[t_i] = \tau + \lambda$  we find the expectation values of the second derivatives,

$$\begin{aligned}E \left[ \frac{\partial^2 \ln L}{\partial \tau^2} \right] &= \frac{n}{(\tau + \lambda)^2} - \frac{2n(\tau + \lambda)}{(\tau + \lambda)^3} = -\frac{n}{(\tau + \lambda)^2}, \\ E \left[ \frac{\partial^2 \ln L}{\partial \lambda^2} \right] &= -\frac{n}{(\tau + \lambda)^2} - \frac{1}{\sigma^2}, \\ E \left[ \frac{\partial^2 \ln L}{\partial \tau \partial \lambda} \right] &= -\frac{n}{(\tau + \lambda)^2}.\end{aligned}$$

The inverse covariance matrix of the estimators is given by

$$V_{ij}^{-1} = -E \left[ \frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right]$$

where here we can take, e.g.,  $\theta_1 = \tau$  and  $\theta_2 = \lambda$ . We are given the formula for the inverse of the corresponding  $2 \times 2$  matrix, and by substituting in the ingredients we find

$$V = \begin{pmatrix} \frac{(\tau + \lambda)^2}{n} + \sigma^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2 \end{pmatrix}$$

which are the same as what was found in (c).