Statistical Methods for Particle Physics
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Solutions to problems on hypothesis tests

1(a) We are given the two pdfs

$$\begin{aligned} f(x|\mathrm{s}) &= 3(x-1)^2 , \\ f(x|\mathrm{b}) &= 3x^2 , \end{aligned}$$

with $0 \le x \le 1$, and we want to select events of type s by requiring $x < x_{\rm cut}$, with $x_{\rm cut} = 0.1$. The probabilities to select events of type s and b are

$$\begin{aligned} P(x < x_{\rm cut}|\mathrm{s}) &= \int_0^{x_{\rm cut}} f(x|\mathrm{s})\,dx = (x-1)^3\Big|_0^{x_{\rm cut}} = (x_{\rm cut}-1)^3 + 1 \\ &= (0.1-1)^3 + 1 = 0.271 \\ P(x < x_{\rm cut}|\mathrm{b}) &= \int_0^{x_{\rm cut}} f(x|\mathrm{b})\,dx = x^3\Big|_0^{x_{\rm cut}} = x_{\rm cut}^3 \\ &= (0.1)^3 = 0.001 \end{aligned}$$

1(b) The signal purity is the probability for an event to be signal given that it is selected. To find this from the available ingredients we apply Bayes' theorem,

$$P(\mathrm{s}|x < x_{\rm cut}) = \frac{P(x < x_{\rm cut}|\mathrm{s})\pi_{\rm s}}{P(x < x_{\rm cut}|\mathrm{s})\pi_{\rm s} + P(x < x_{\rm cut}|\mathrm{b})\pi_{\rm b}} = \frac{(1 + (x_{\rm cut}-1)^3)\pi_{\rm s}}{(1 + (x_{\rm cut}-1)^3)\pi_{\rm s} + x_{\rm cut}^3\pi_{\rm b}} ,$$

where $\pi_{\rm s} = 0.01$ and $\pi_{\rm b} = 0.99$ are the given prior probabilities. Plugging in the numbers gives

$$P(\mathrm{s}|x < x_{\rm cut}) = \frac{0.271 \times 0.01}{0.271 \times 0.01 + 0.001 \times 0.99} = 0.732 ,$$

1(c) For an event with an observed value of x, the probability that it is background is again given by Bayes' theorem,

$$P(\mathrm{b}|x) = \frac{f(x|\mathrm{b})\pi_{\rm b}}{f(x|\mathrm{b})\pi_{\rm b} + f(x|\mathrm{s})\pi_{\rm s}} = \frac{x^2\pi_{\rm b}}{x^2\pi_{\rm b} + (x-1)^2\pi_{\rm s}} = \frac{0.05^2 \times 0.99}{0.05^2 \times 0.99 + 0.95^2 \times 0.01} = 0.215 .$$

1(d) The pdf $f(x|\mathrm{b}) = 3x^2$ is concentrated towards one, and $f(x|\mathrm{s}) = 3(x-1)^2$ towards zero. So if we observe $x = 0.05$, then values of x less than this represent less compatibility with $f(x|\mathrm{b})$. Therefore the p-value of the background hypothesis can be obtained as

$$p = \int_0^x f(x'|\mathrm{b})\,dx' = \int_0^x 3x'^2\,dx' = x^3 = 0.05^3 = 1.25 \times 10^{-4} .$$

This is not the same as the probability for the event to be of type b, but rather the probability, assuming b, to observe x with equal or lesser compatibility with b than what was found with the actual data. Unlike the probability $P(\mathrm{b}|x)$ found in (c), the p-value is independent of the prior probability for the event to be of type b.

1(f) We are now given two joint pdfs for x and y,

$$\begin{aligned} f(x,y|\mathrm{s}) &= 6(x-1)^2 y \ , \\ f(x,y|\mathrm{b}) &= 6x^2(1-y) \ , \end{aligned}$$

with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. According to the Neyman-Pearson lemma, the test statistic that gives the highest power for a given significance level test (in this case equivalent to having the highest signal purity for a given efficiency), is given by the likelihood ratio

$$t(x) = \frac{f(x,y|\mathrm{s})}{f(x,y|\mathrm{b})} = \frac{(x-1)^2 y}{x^2(1-y)} \ .$$