Problems in Parameter Estimation
INFN School of Statistics
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1. An experiment yields n time values, $t_1,\ldots,t_n$, and a calibration value y, all of which are independent. The time measurements are all exponentially distributed with a mean of $\tau+\lambda$ and the calibration measurement, y, follows a Gaussian distribution with a mean λ and standard deviation σ. Suppose that σ is known and we want to estimate τ and λ.

(a) Write down the likelihood function for τ and λ, and show that the Maximum Likelihood (ML) estimators for these parameters are

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} t_i - y$$

and

$$\hat{\lambda} = y .$$

(b) Find the variances of $\hat{\tau}$ and $\hat{\lambda}$, and the covariance $\mathrm{cov}[\hat{\tau},\hat{\lambda}]$. Use the fact that the variance of an exponentially distributed variable is equal to the square of its mean.

(c) Show using a sketch how a contour of constant log-likelihood can be used to determine the standard deviations of $\hat{\tau}$ and $\hat{\lambda}$.

Explain qualitatively how you would expect the variance of $\hat{\tau}$ to be different if the parameter λ were to be known exactly.

(d) Show that the (co)variances of $\hat{\tau}$ and $\hat{\lambda}$ obtained from the matrix of second derivatives of the log-likelihood are the same as those found in (c). Use the fact that the inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2. A measurement yields two values, n_{a} and n_b, which are independent and Poisson distributed with mean values ν_{a} and ν_{b}, respectively. From these parameters we define the total mean, ν, and asymmetry parameter, α, as

$$\nu = \nu_{\mathrm{a}} + \nu_{\mathrm{b}} \ ,$$

$$\alpha = \frac{\nu_{\mathrm{a}} - \nu_{\mathrm{b}}}{\nu_{\mathrm{a}} + \nu_{\mathrm{b}}} \ .$$

Recall that the Poisson distribution is $P(n;\nu) = \nu^n e^{-\nu} / n!$.

(a) Write down the likelihood function in terms of ν and α and find the Maximum Likelihood (ML) estimators for these parameters.

(b) Estimate the variance of $\hat{\alpha}$ as a function of ν and α using error propagation.

(c) Consider the Bayesian approach to inference about ν and α, and suppose we take the prior pdf for the parameters to be

$$\pi(\nu,\alpha) \propto 1/\sqrt{\nu} \ .$$

Find as a proportionality the joint posterior pdf for ν and α given n_{a} and n_b.

Show that ν and α are independent, and that the marginal posterior pdfs follow

$$p(\nu \mid n_{\mathrm{a}}, n_{\mathrm{b}}) \propto \nu^{(n_{\mathrm{a}} + n_{\mathrm{b}} - 1/2)} e^{-\nu} \ ,$$

$$p(\alpha \mid n_{\mathrm{a}}, n_{\mathrm{b}}) \propto (1+\alpha)^{n_{\mathrm{a}}} (1-\alpha)^{n_{\mathrm{b}}} \ .$$

(d) Find the posterior modes for ν and α. Comment on how these relate to the corresponding ML estimators.