Statistical Methods for Particle Physics
Lecture 1: introduction to frequentist statistics

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Statistical Inference for Astro and Particle Physics Workshop
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Outline for Monday – Thursday

(GC = Glen Cowan, KC = Kyle Cranmer)

Monday 9 March

GC: probability, random variables and related quantities

KC: parameter estimation, bias, variance, max likelihood

Tuesday 10 March

KC: building statistical models, nuisance parameters

GC: hypothesis tests I, *p*-values, multivariate methods

Wednesday 11 March

KC: hypothesis tests 2, composite hyp., Wilks', Wald's thm.

GC: asympotics 1, Asimov data set, sensitivity

Thursday 12 March:

KC: confidence intervals, asymptotics 2

GC: unfolding

Some statistics books, papers, etc.

G. Cowan, *Statistical Data Analysis*, Clarendon, Oxford, 1998

R.J. Barlow, *Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences*, Wiley, 1989

Ilya Narsky and Frank C. Porter, *Statistical Analysis Techniques in Particle Physics*, Wiley, 2014.

L. Lyons, *Statistics for Nuclear and Particle Physics*, CUP, 1986

F. James., *Statistical and Computational Methods in Experimental Physics*, 2nd ed., World Scientific, 2006

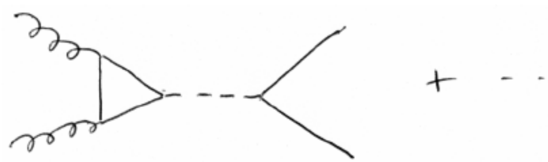
S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1998 (with program library on CD)

J. Beringer et al. (Particle Data Group), *Review of Particle Physics*, Phys. Rev. D86, 010001 (2012) ; see also **`pdg.lbl.gov`** sections on probability, statistics, Monte Carlo

Theory ↔ Statistics ↔ Experiment

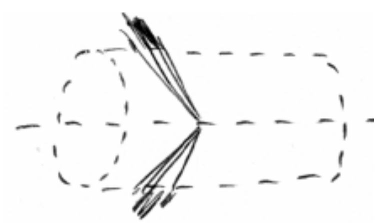
Theory (model, hypothesis):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi + \cdots$$

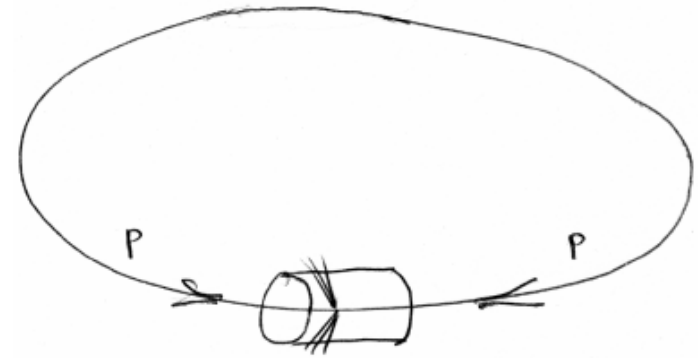


$$\sigma = \frac{G_F\,\alpha_s^2\,m_H^2}{288\sqrt{2\pi}} \times \sim$$

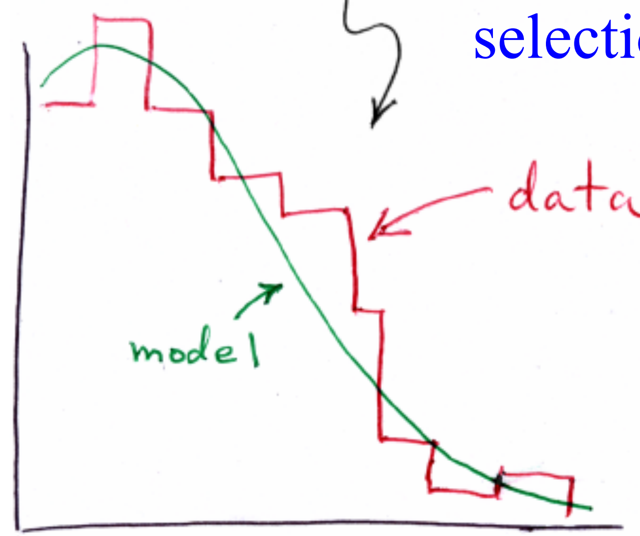
+ simulation of detector and cuts



Experiment:



+ data selection



Data analysis in particle physics

Observe events (e.g., pp collisions) and for each, measure a set of characteristics:

particle momenta, number of muons, energy of jets,...

Compare observed distributions of these characteristics to predictions of theory. From this, we want to:

Estimate the free parameters of the theory: $m_H = 125.4$

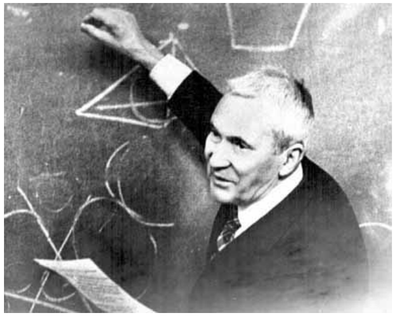
Quantify the uncertainty in the estimates: $\pm\ 0.4$ GeV

Assess how well a given theory stands in agreement with the observed data: 0^+ good, 2^+ bad

To do this we need a clear definition of PROBABILITY

A definition of probability

Consider a set S with subsets A, B, ...



For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$

Kolmogorov axioms (1933)

From these axioms we can derive further properties, e.g.

$$P(\overline{A}) = 1 - P(A)$$
$$P(A \cup \overline{A}) = 1$$
$$P(\emptyset) = 0$$
$$\text{if } A \subset B, \text{ then } P(A) \leq P(B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability, independence

Also define conditional probability of A given B (with $P(B) \neq 0$):

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

E.g. rolling dice: $P(n < 3 \,|\, n \text{ even}) = \frac{P((n<3) \cap n \text{ even})}{P(\text{even})} = \frac{1/6}{3/6} = \frac{1}{3}$

Subsets A, B independent if: $P(A \cap B) = P(A)P(B)$

If A, B independent, $P(A|B) = \dfrac{P(A)P(B)}{P(B)} = P(A)$

N.B. do not confuse with disjoint subsets, i.e., $A \cap B = \emptyset$

Interpretation of probability

I. Relative frequency

A, B, ... are outcomes of a repeatable experiment

$$P(A) = \lim_{n \to \infty} \frac{\text{times outcome is } A}{n}$$

cf. quantum mechanics, particle scattering, radioactive decay...

II. Subjective probability

A, B, ... are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

- Both interpretations consistent with Kolmogorov axioms.
- In particle physics frequency interpretation often most useful, but subjective probability can provide more natural treatment of non-repeatable phenomena:

systematic uncertainties, probability that Higgs boson exists,...

Bayes' theorem

From the definition of conditional probability we have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

but $P(A \cap B) = P(B \cap A)$, so

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Bayes' theorem

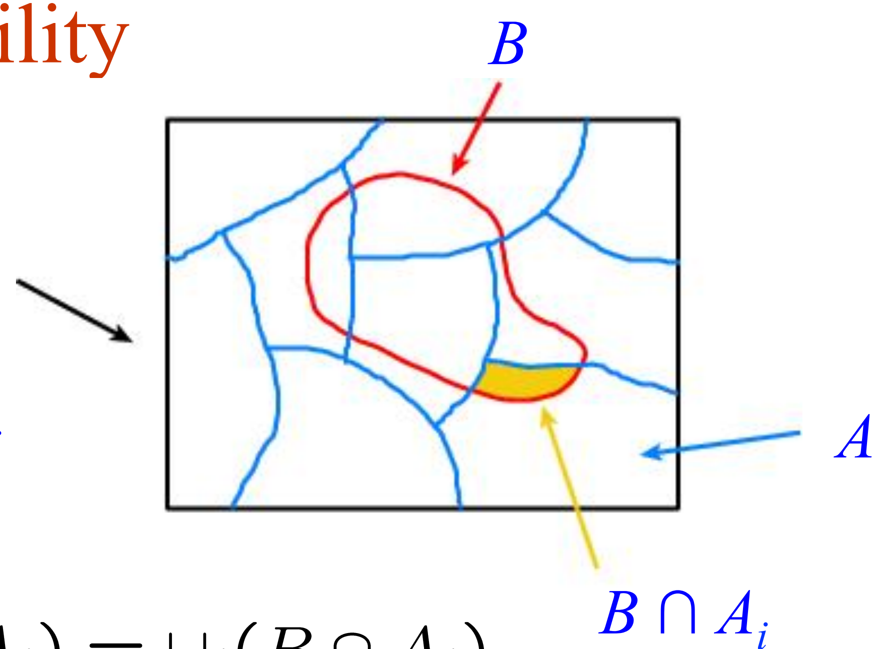
First published (posthumously) by the Reverend Thomas Bayes (1702−1761)

An essay towards solving a problem in the doctrine of chances, Philos. Trans. R. Soc. **53** (1763) 370; reprinted in Biometrika, **45** (1958) 293.

The law of total probability

Consider a subset B of the sample space S,

divided into disjoint subsets A_i such that $\cup_i A_i = S$,



$\rightarrow \quad B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i),$

$\rightarrow \quad P(B) = P(\cup_i (B \cap A_i)) = \sum_i P(B \cap A_i)$

$\rightarrow \quad P(B) = \sum_i P(B|A_i) P(A_i)$ law of total probability

Bayes' theorem becomes

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}}$$

An example using Bayes' theorem

Suppose the probability (for anyone) to have a disease D is:

$$P(\mathrm{D}) = 0.001$$
$$P(\text{no D}) = 0.999$$

← prior probabilities, i.e., before any test carried out

Consider a test for the disease: result is + or −

$$P(+|\mathrm{D}) = 0.98$$
$$P(-|\mathrm{D}) = 0.02$$

← probabilities to (in)correctly identify a person with the disease

$$P(+|\text{no D}) = 0.03$$
$$P(-|\text{no D}) = 0.97$$

← probabilities to (in)correctly identify a healthy person

Suppose your result is +. How worried should you be?

Bayes' theorem example (cont.)

The probability to have the disease given a + result is

$$p(\mathrm{D}|+) = \frac{P(+|\mathrm{D})P(\mathrm{D})}{P(+|\mathrm{D})P(\mathrm{D}) + P(+|\text{no D})P(\text{no D})}$$

$$= \frac{0.98 \times 0.001}{0.98 \times 0.001 + 0.03 \times 0.999}$$

$$= 0.032 \qquad \leftarrow \text{posterior probability}$$

i.e. you're probably OK!

Your viewpoint: my degree of belief that I have the disease is 3.2%.

Your doctor's viewpoint: 3.2% of people like this have the disease.

Frequentist Statistics − general philosophy

In frequentist statistics, probabilities are associated only with the data, i.e., outcomes of repeatable observations (shorthand: $\vec{x}$).

Probability = limiting frequency

Probabilities such as

P (Higgs boson exists),
P $(0.117 < \alpha_s < 0.121)$,

etc. are either 0 or 1, but we don't know which.

The tools of frequentist statistics tell us what to expect, under the assumption of certain probabilities, about hypothetical repeated observations.

A hypothesis is is preferred if the data are found in a region of high predicted probability (i.e., where an alternative hypothesis predicts lower probability).

Bayesian Statistics − general philosophy

In Bayesian statistics, use subjective probability for hypotheses:

probability of the data assuming hypothesis H (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H)\,dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

Bayes' theorem has an "if-then" character: If your prior probabilities were $\pi(H)$, then it says how these probabilities should change in the light of the data.

No general prescription for priors (subjective!)

Random variables and probability density functions

A random variable is a numerical characteristic assigned to an element of the sample space; can be discrete or continuous.

Suppose outcome of experiment is continuous value x

$$P(x \text{ found in } [x, x + dx]) = f(x)\, dx$$

→ $f(x)$ = probability density function (pdf)

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

x must be somewhere

Or for discrete outcome x_i with e.g. i = 1, 2, ... we have

$$P(x_i) = p_i$$

probability mass function

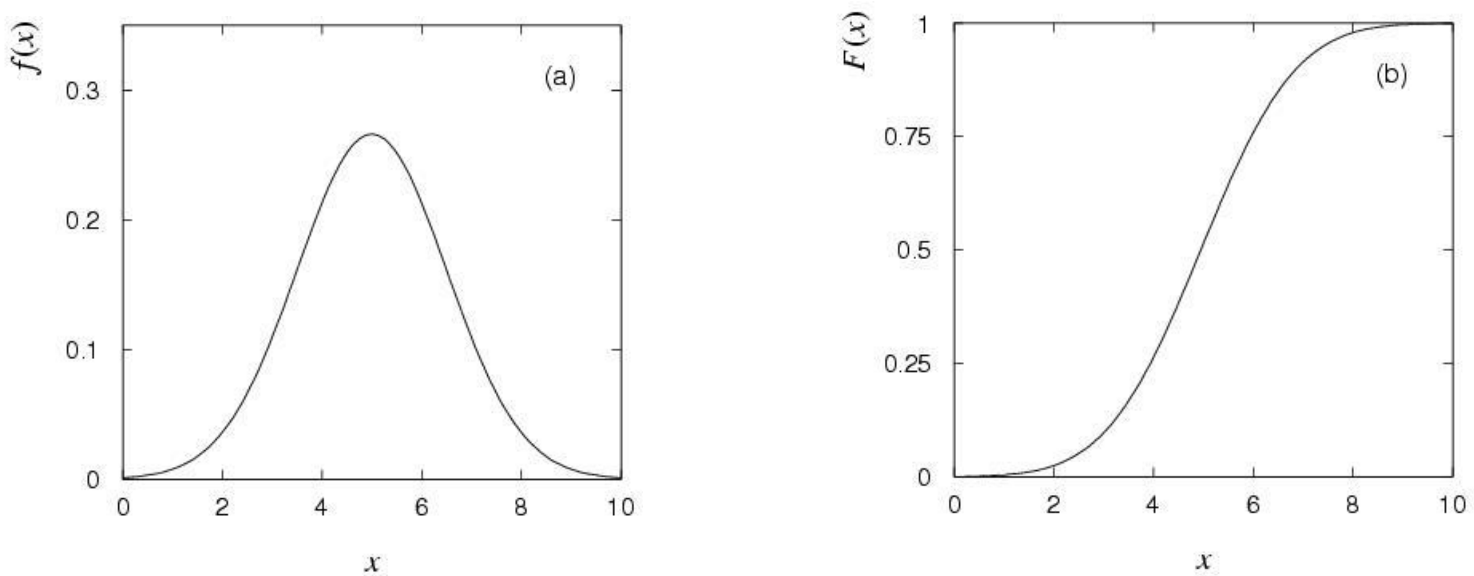
$$\sum_i P(x_i) = 1$$

x must take on one of its possible values

Cumulative distribution function

Probability to have outcome less than or equal to x is

$$\int_{-\infty}^{x} f(x')\,dx' \equiv F(x)$$ cumulative distribution function



Alternatively define pdf with $f(x) = \dfrac{\partial F(x)}{\partial x}$

Other types of probability densities

Outcome of experiment characterized by several values, e.g. an n-component vector, $(x_1, \ldots x_n)$

→ joint pdf $f(x_1, \ldots, x_n)$

Sometimes we want only pdf of some (or one) of the components

→ marginal pdf $f_1(x_1) = \int \cdots \int f(x_1, \ldots, x_n)\, dx_2 \ldots dx_n$

x_1, x_2 independent if $f(x_1, x_2) = f_1(x_1) f_2(x_2)$

Sometimes we want to consider some components as constant

→ conditional pdf $g(x_1|x_2) = \dfrac{f(x_1, x_2)}{f_2(x_2)}$

Distribution, likelihood, model

Suppose the outcome of a measurement is x. (e.g., a number of events, a histogram, or some larger set of numbers).

The probability density (or mass) function or ‘distribution’ of x, which may depend on parameters θ, is:

$P(x|\theta)$ (Independent variable is x; θ is a constant.)

If we evaluate $P(x|\theta)$ with the observed data and regard it as a function of the parameter(s), then this is the likelihood:

$L(\theta) = P(x|\theta)$ (Data x fixed; treat L as function of θ.)

We will use the term ‘model’ to refer to the full function $P(x|\theta)$ that contains the dependence both on x and θ.

Bayesian use of the term ‘likelihood’

We can write Bayes theorem as

$$p(\theta|x) = \frac{L(x|\theta)\pi(\theta)}{\int L(x|\theta)\pi(\theta)\,d\theta}$$

where $L(x|\theta)$ is the likelihood. It is the probability for x given θ, evaluated with the observed x, and viewed as a function of θ.

Bayes’ theorem only needs $L(x|\theta)$ evaluated with a given data set (the ‘likelihood principle’).

For frequentist methods, in general one needs the full model.

For some approximate frequentist methods, the likelihood is enough.

The likelihood function for i.i.d.*. data

* i.i.d. = independent and identically distributed

Consider n independent observations of x: $x_1, \ldots, x_n$, where x follows $f(x; \theta)$. The joint pdf for the whole data sample is:

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

In this case the likelihood function is

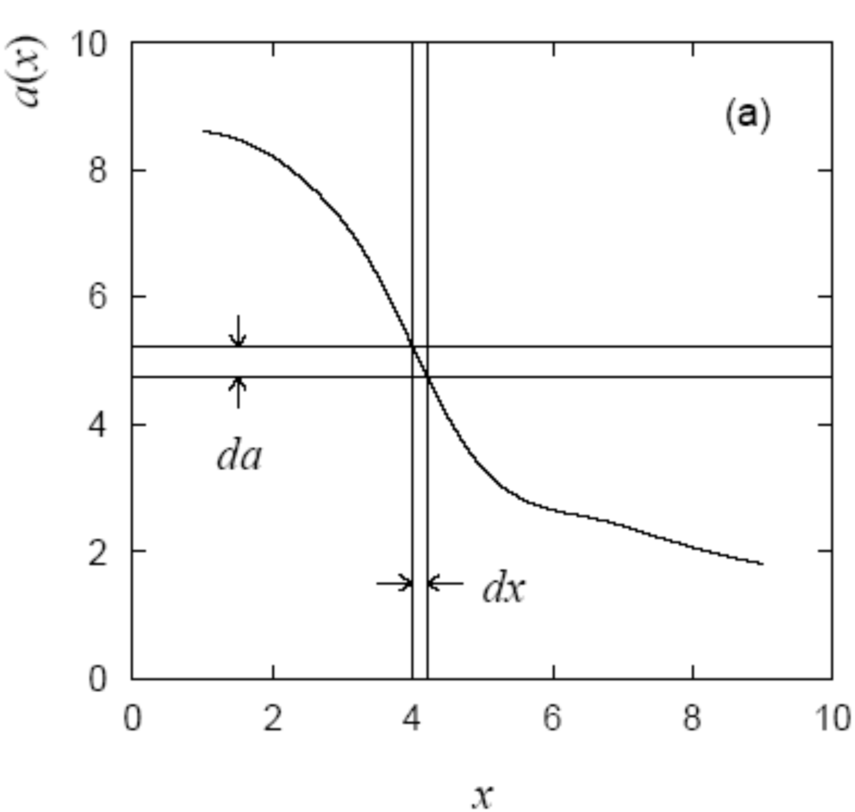
$$L(\vec{\theta}) = \prod_{i=1}^{n} f(x_i; \vec{\theta}) \qquad (x_i \text{ constant})$$

Functions of a random variable

A function of a random variable is itself a random variable.

Suppose x follows a pdf $f(x)$, consider a function $a(x)$.

What is the pdf $g(a)$?



$$g(a)\,da = \int_{dS} f(x)\,dx$$

dS = region of x space for which a is in $[a, a+da]$.

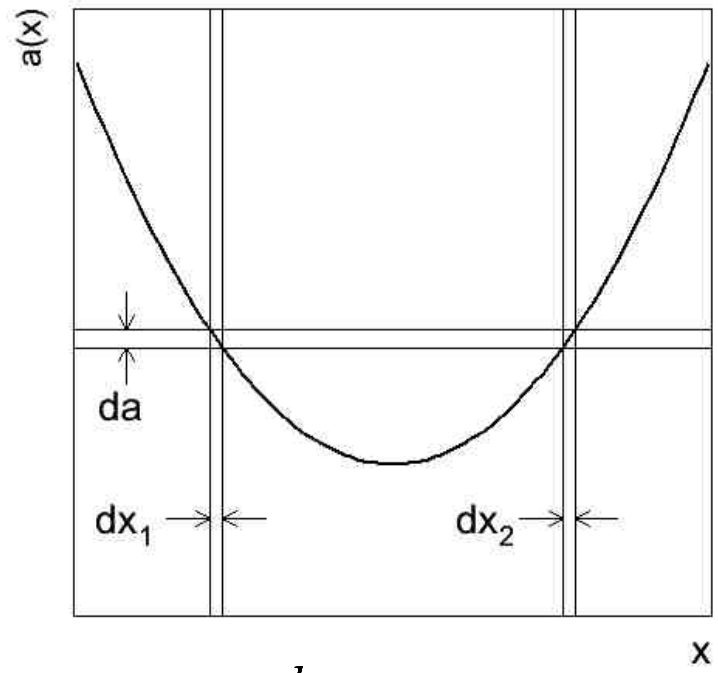
For one-variable case with unique inverse this is simply

$$g(a)\,da = f(x)\,dx$$

$$\rightarrow \quad g(a) = f(x(a)) \left| \frac{dx}{da} \right|$$

Functions without unique inverse

If inverse of $a(x)$ not unique, include all dx intervals in dS which correspond to da:



Example: $$a = x^2, \;\; x = \pm\sqrt{a}, \;\; dx = \pm\frac{da}{2\sqrt{a}}\,.$$

$$dS = \left[\sqrt{a}, \sqrt{a} + \frac{da}{2\sqrt{a}}\right] \cup \left[-\sqrt{a} - \frac{da}{2\sqrt{a}}, -\sqrt{a}\right]$$

$$g(a) = \frac{f(\sqrt{a})}{2\sqrt{a}} + \frac{f(-\sqrt{a})}{2\sqrt{a}}$$

Functions of more than one r.v.

Consider r.v.s $\vec{x} = (x_1, \ldots, x_n)$ and a function $a(\vec{x})$.

$$g(a')da' = \int \ldots \int_{dS} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

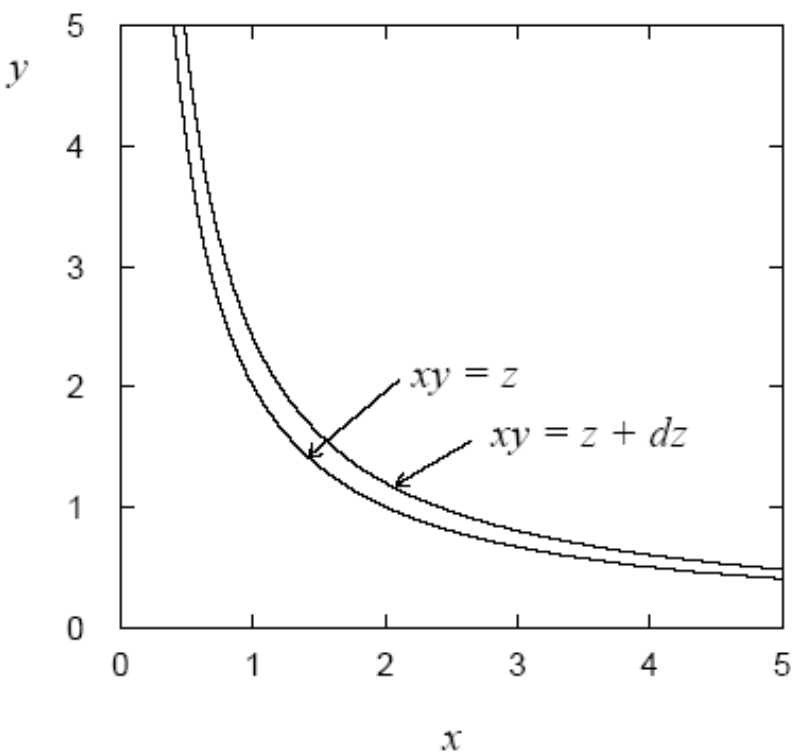
dS = region of x-space between (hyper)surfaces defined by

$$a(\vec{x}) = a', \; a(\vec{x}) = a' + da'$$

Functions of more than one r.v. (2)

Example: r.v.s x, $y > 0$ follow joint pdf $f(x,y)$,

consider the function $z = xy$. What is $g(z)$?



$$g(z)\,dz = \int \ldots \int_{dS} f(x,y)\,dx\,dy$$

$$= \int_0^\infty dx \int_{z/x}^{(z+dz)/x} f(x,y)\,dy$$

$$\rightarrow \quad g(z) = \int_0^\infty f(x, \frac{z}{x}) \frac{dx}{x}$$

$$= \int_0^\infty f(\frac{z}{y}, y) \frac{dy}{y}$$

(Mellin convolution)

More on transformation of variables

Consider a random vector $\vec{x} = (x_1, \ldots, x_n)$ with joint pdf $f(\vec{x})$.

Form n linearly independent functions $\vec{y}(\vec{x}) = (y_1(\vec{x}), \ldots, y_n(\vec{x}))$

for which the inverse functions $x_1(\vec{y}), \ldots, x_n(\vec{y})$ exist.

Then the joint pdf of the vector of functions is $g(\vec{y}) = |J| f(\vec{x})$

where J is the Jacobian determinant:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \vdots \\ & & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

For e.g. $g_1(y_1)$ integrate $g(\vec{y})$ over the unwanted components.

Expectation values

Consider continuous r.v. x with pdf $f(x)$.

Define expectation (mean) value as $E[x] = \int x\, f(x)\, dx$

Notation (often): $E[x] = \mu$ ~ "centre of gravity" of pdf.

For a function $y(x)$ with pdf $g(y)$,

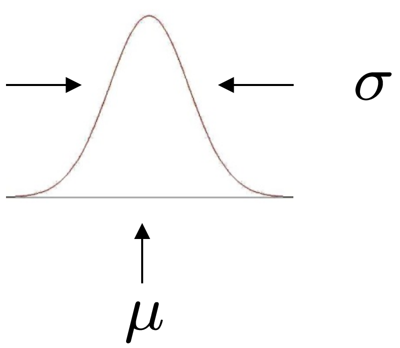
$$E[y] = \int y\, g(y)\, dy = \int y(x) f(x)\, dx \qquad \text{(equivalent)}$$

Variance: $V[x] = E[x^2] - \mu^2 = E[(x-\mu)^2]$

Notation: $V[x] = \sigma^2$

Standard deviation: $\sigma = \sqrt{\sigma^2}$

σ ~ width of pdf, same units as x.



Quantile, median, mode

The *quantile* or α-point x_α of a random variable x is inverse of the cumulative distribution ,i.e., the value of x such that

$$x_\alpha = F^{-1}(\alpha)$$

The special case $x_{1/2}$ is called the *median*, med[x], i.e., the value of x such that $P(x \leq x_{1/2}) = 1/2$.

For a monotonic transformation $x \to y(x)$, one has $y_\alpha = y(x_\alpha)$.

The *mode* of a random variable is the value is the value with the maximum probability, or at the maximum of the pdf.

For a nonlinear transformation $x \to y(x)$, in general $\text{mode}[y] \neq y(\text{mode}[x])$

Moments of a random variable

The n^{th} algebraic moment of (continuous) x is defined as:

$$E[x^n] = \int_{-\infty}^{\infty} x^n f(x) dx = \mu'_n$$

First (n=1) algebraic moment is the mean: $\mu = \mu'_1$

The n^{th} cemtral moment of x is defined as:

$$E[(x - E[x])^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx = \mu_n$$

Second central moment is the variance: $V[x] = \sigma^2 = \mu_2$

Covariance and correlation

Define covariance cov[x,y] (also use matrix notation V_{xy}) as

$$\mathrm{cov}[x,y] = E[xy] - \mu_x \mu_y = E[(x-\mu_x)(y-\mu_y)]$$

Correlation coefficient (dimensionless) defined as

$$\rho_{xy} = \frac{\mathrm{cov}[x,y]}{\sigma_x \sigma_y}$$

If x, y, independent, i.e., $f(x,y) = f_x(x) f_y(y)$, then

$$E[xy] = \int \int xy\, f(x,y)\, dx dy = \mu_x \mu_y$$

$\rightarrow$ $\mathrm{cov}[x,y] = 0$ x and y, ‘uncorrelated’

N.B. converse not always true.

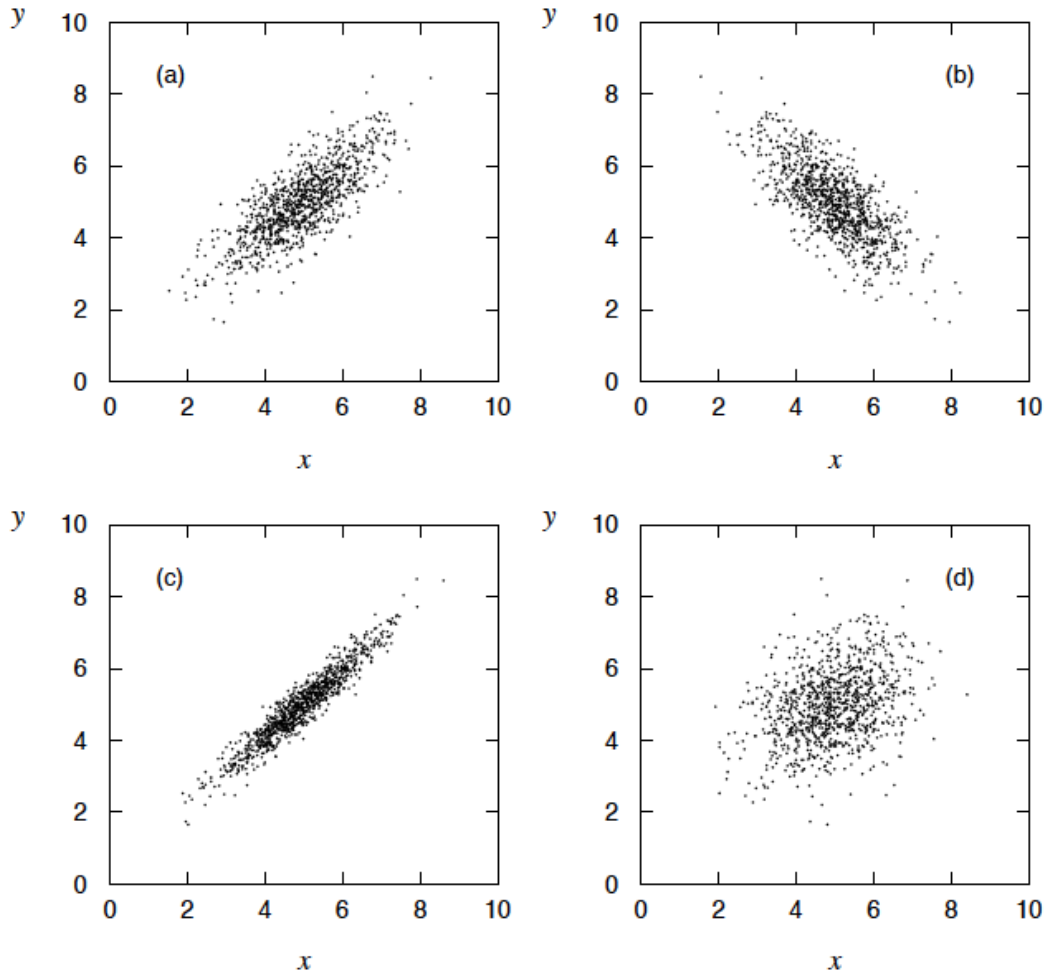
Correlation (cont.)

$\rho = 0.75$

$\rho = -0.75$

$\rho = 0.95$

$\rho = 0.25$



Error propagation

Suppose we measure a set of values $\vec{x} = (x_1, \ldots, x_n)$

and we have the covariances $V_{ij} = \text{cov}[x_i, x_j]$

which quantify the measurement errors in the x_i.

Now consider a function $y(\vec{x})$.

What is the variance of $y(\vec{x})$?

The hard way: use joint pdf $f(\vec{x})$ to find the pdf $g(y)$,

then from $g(y)$ find $V[y] = E[y^2] - (E[y])^2$.

Often not practical, $f(\vec{x})$ may not even be fully known.

Error propagation (2)

Suppose we had $\vec{\mu} = E[\vec{x}]$

in practice only estimates given by the measured $\vec{x}$

Expand $y(\vec{x})$ to 1st order in a Taylor series about $\vec{\mu}$

$$y(\vec{x}) \approx y(\vec{\mu}) + \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} (x_i - \mu_i)$$

To find $V[y]$ we need $E[y^2]$ and $E[y]$.

$$E[y(\vec{x})] \approx y(\vec{\mu}) \quad \text{since} \quad E[x_i - \mu_i] = 0$$

Error propagation (3)

$$E[y^2(\vec{x})] \approx y^2(\vec{\mu}) \; + \; 2y(\vec{\mu}) \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} E[x_i - \mu_i]$$

$$+E\left[\left(\sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} (x_i - \mu_i) \right) \left(\sum_{j=1}^{n} \left[\frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} (x_j - \mu_j) \right) \right]$$

$$= y^2(\vec{\mu}) + \sum_{i,j=1}^{n} \left[\frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}$$

Putting the ingredients together gives the variance of $y(\vec{x})$

$$\boxed{\sigma_y^2 \approx \sum_{i,j=1}^{n} \left[\frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}}$$

Error propagation (4)

If the x_i are uncorrelated, i.e., $V_{ij} = \sigma_i^2 \delta_{ij}$, then this becomes

$$\sigma_y^2 \approx \sum_{i=1}^{n} \left[\frac{\partial y}{\partial x_i} \right]^2_{\vec{x}=\vec{\mu}} \sigma_i^2$$

Similar for a set of m functions $\vec{y}(\vec{x}) = (y_1(\vec{x}), \ldots, y_m(\vec{x}))$

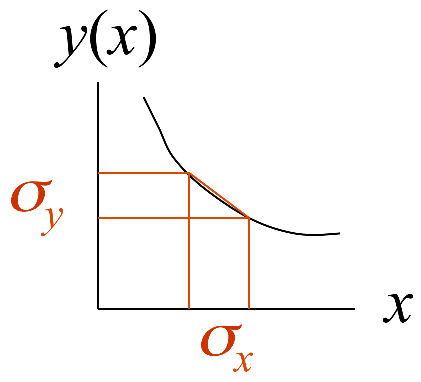
$$\boxed{U_{kl} = \mathrm{cov}[y_k, y_l] \approx \sum_{i,j=1}^{n} \left[\frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}}$$

or in matrix notation $U = AVA^T$, where

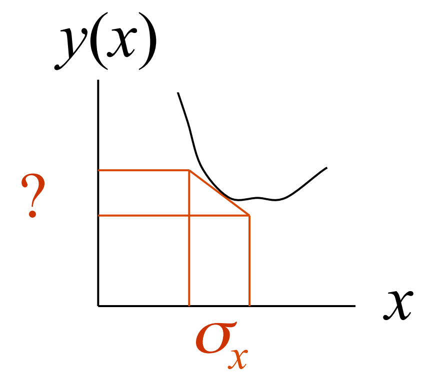
$$A_{ij} = \left[\frac{\partial y_i}{\partial x_j} \right]_{\vec{x}=\vec{\mu}}$$

Error propagation (5)

The ‘error propagation’ formulae tell us the covariances of a set of functions $\vec{y}(\vec{x}) = (y_1(\vec{x}), \ldots, y_m(\vec{x}))$ in terms of the covariances of the original variables.



Limitations: exact only if $\vec{y}(\vec{x})$ linear.
Approximation breaks down if function nonlinear over a region comparable in size to the σ_i.



N.B. We have said nothing about the exact pdf of the x_i, e.g., it doesn’t have to be Gaussian.

Error propagation – special cases

$$y = x_1 + x_2 \quad \rightarrow \quad \sigma_y^2 = \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{cov}[x_1, x_2]$$

$$y = x_1 x_2 \quad \rightarrow \quad \frac{\sigma_y^2}{y^2} = \frac{\sigma_1^2}{x_1^2} + \frac{\sigma_2^2}{x_2^2} + 2\frac{\mathrm{cov}[x_1, x_2]}{x_1 x_2}$$

That is, if the x_i are uncorrelated:

add errors quadratically for the sum (or difference),

add relative errors quadratically for product (or ratio).

But correlations can change this completely...

Error propagation − special cases (2)

Consider $y = x_1 - x_2$ with

$$\mu_1 = \mu_2 = 10, \quad \sigma_1 = \sigma_2 = 1, \quad \rho = \frac{\mathrm{cov}[x_1, x_2]}{\sigma_1 \sigma_2} = 0\,.$$

$$V[y] = 1^2 + 1^2 = 2, \rightarrow \sigma_y = 1.4$$

Now suppose $\rho = 1$. Then

$$V[y] = 1^2 + 1^2 - 2 = 0, \rightarrow \sigma_y = 0$$

i.e. for 100% correlation, error in difference → 0.

Some distributions

Distribution/pdf	Example use in HEP
Binomial	Branching ratio
Multinomial	Histogram with fixed N
Poisson	Number of events found
Uniform	Monte Carlo method
Exponential	Decay time
Gaussian	Measurement error
Chi-square	Goodness-of-fit
Cauchy	Mass of resonance
Landau	Ionization energy loss
Beta	Prior pdf for efficiency
Gamma	Sum of exponential variables
Student's t	Resolution function with adjustable tails

Binomial distribution

Consider N independent experiments (Bernoulli trials):

outcome of each is 'success' or 'failure',

probability of success on any given trial is p.

Define discrete r.v. n = number of successes ($0 \le n \le N$).

Probability of a specific outcome (in order), e.g. 'ssfsf' is

$$pp(1-p)p(1-p) = p^n (1-p)^{N-n}$$

But order not important; there are $\dfrac{N!}{n!(N-n)!}$

ways (permutations) to get n successes in N trials, total probability for n is sum of probabilities for each permutation.

Binomial distribution (2)

The binomial distribution is therefore

$$f(n; N, p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

random variable

parameters

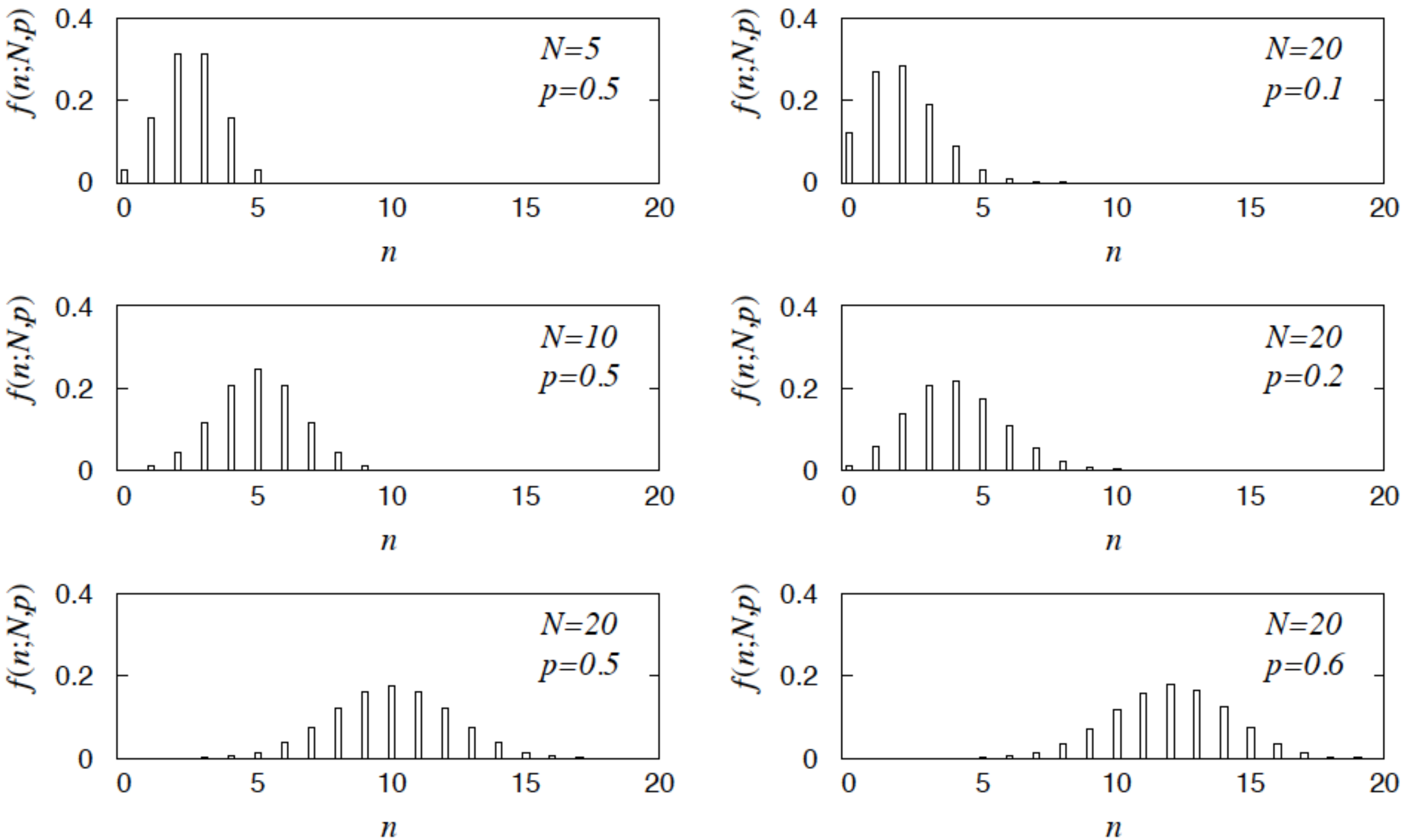
For the expectation value and variance we find:

$$E[n] = \sum_{n=0}^{N} n f(n; N, p) = Np$$

$$V[n] = E[n^2] - (E[n])^2 = Np(1-p)$$

Binomial distribution (3)

Binomial distribution for several values of the parameters:



Example: observe N decays of $W^{\pm}$, the number n of which are $W \to \mu\nu$ is a binomial r.v., p = branching ratio.

Multinomial distribution

Like binomial but now m outcomes instead of two, probabilities are

$$\vec{p} = (p_1, \ldots, p_m) , \quad \text{with } \sum_{i=1}^{m} p_i = 1 .$$

For N trials we want the probability to obtain:

n_1 of outcome 1,
n_2 of outcome 2,
…
n_m of outcome m.

This is the multinomial distribution for $\vec{n} = (n_1, \ldots, n_m)$

$$f(\vec{n}; N, \vec{p}) = \frac{N!}{n_1! n_2! \cdots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

Multinomial distribution (2)

Now consider outcome i as 'success', all others as 'failure'.

→ all n_i individually binomial with parameters N, p_i

$$E[n_i] = Np_i, \quad V[n_i] = Np_i(1 - p_i) \quad \text{for all } i$$

One can also find the covariance to be

$$V_{ij} = Np_i(\delta_{ij} - p_j)$$

Example: $\vec{n} = (n_1, \ldots, n_m)$ represents a histogram

with m bins, N total entries, all entries independent.

Poisson distribution

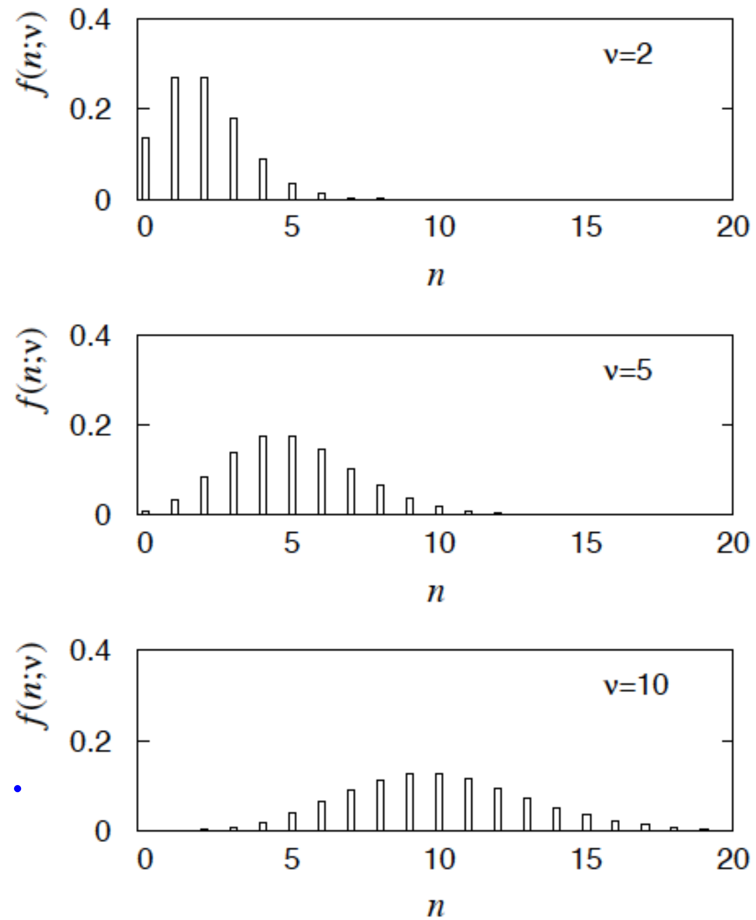
Consider binomial n in the limit

$$N \to \infty, \qquad p \to 0, \qquad E[n] = Np \to \nu \; .$$

$\to$ n follows the Poisson distribution:

$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu} \qquad (n \geq 0)$$

$$E[n] = \nu \, , \qquad V[n] = \nu \; .$$



Example: number of scattering events n with cross section σ found for a fixed integrated luminosity, with $\nu = \sigma \int L \, dt$.

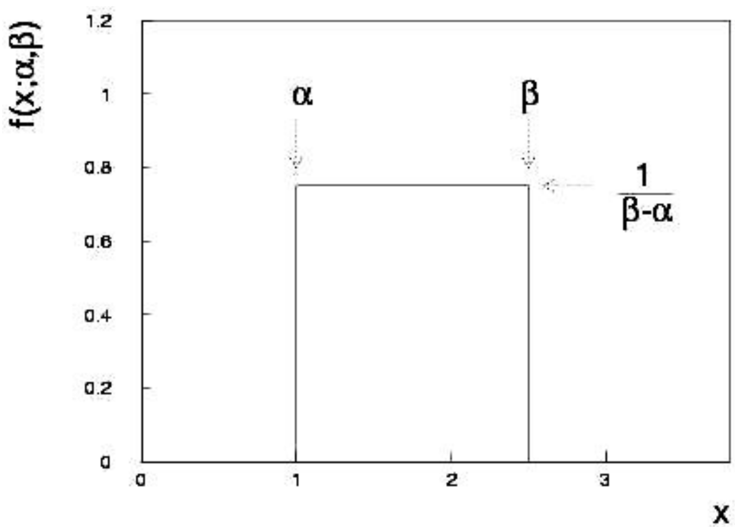
Uniform distribution

Consider a continuous r.v. x with $-\infty < x < \infty$. Uniform pdf is:

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{2}(\alpha+\beta)$$

$$V[x] = \frac{1}{12}(\beta-\alpha)^2$$



N.B. For any r.v. x with cumulative distribution $F(x)$,
$y = F(x)$ is uniform in [0,1].

Example: for $\pi^0 \to \gamma\gamma$, E_γ is uniform in $[E_{\min}, E_{\max}]$, with

$$E_{\min} = \frac{1}{2}E_\pi(1-\beta)\,, \qquad E_{\max} = \frac{1}{2}E_\pi(1+\beta)$$

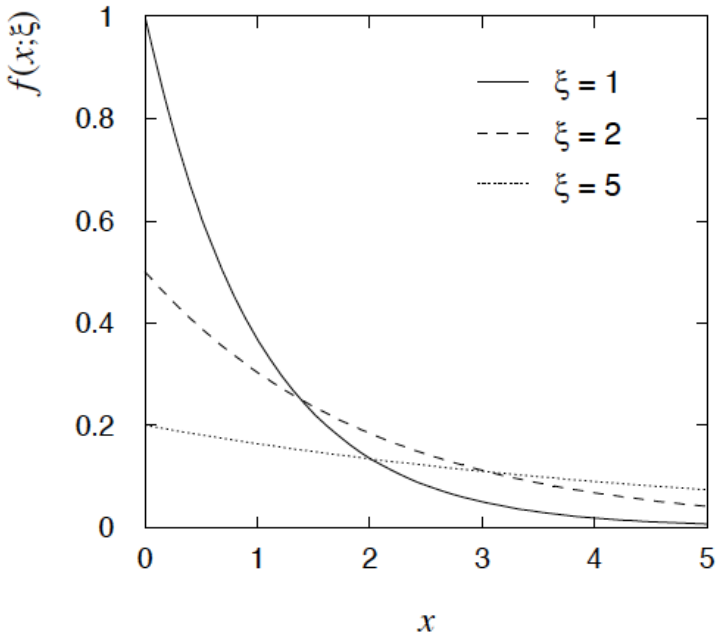
Exponential distribution

The exponential pdf for the continuous r.v. *x* is defined by:

$$f(x;\xi) = \begin{cases} \frac{1}{\xi} e^{-x/\xi} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \xi$$

$$V[x] = \xi^2$$



Example: proper decay time *t* of an unstable particle

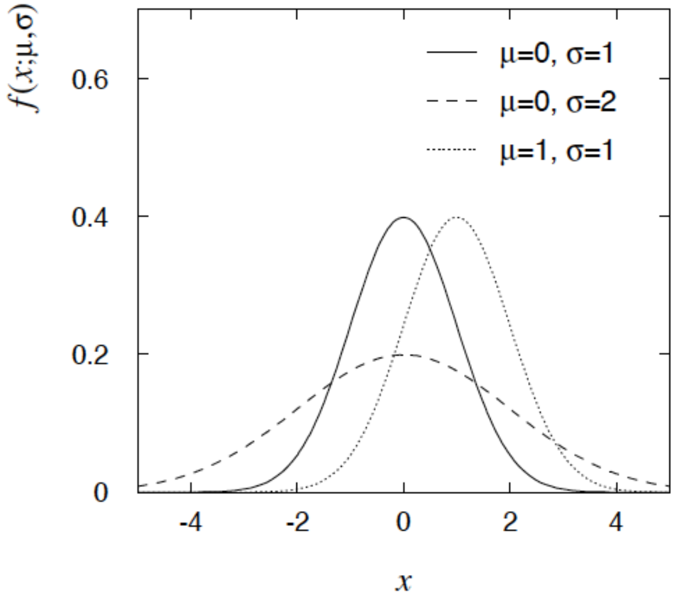
$$f(t;\tau) = \frac{1}{\tau} e^{-t/\tau} \qquad (\tau = \text{mean lifetime})$$

Lack of memory (unique to exponential): $f(t - t_0 | t \geq t_0) = f(t)$

Gaussian distribution

The Gaussian (normal) pdf for a continuous r.v. x is defined by:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



$E[x] = \mu$

$V[x] = \sigma^2$

(N.B. often μ, σ^2 denote mean, variance of any r.v., not only Gaussian.)

Special case: $\mu = 0$, $\sigma^2 = 1$ ('standard Gaussian'):

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(x')\,dx'$$

If $y \sim$ Gaussian with μ, σ^2, then $x = (y - \mu)/\sigma$ follows $\varphi(x)$.

Gaussian pdf and the Central Limit Theorem

The Gaussian pdf is so useful because almost any random variable that is a sum of a large number of small contributions follows it. This follows from the Central Limit Theorem:

For n independent r.v.s x_i with finite variances σ_i^2, otherwise arbitrary pdfs, consider the sum

$$y = \sum_{i=1}^{n} x_i$$

In the limit $n \to \infty$, y is a Gaussian r.v. with

$$E[y] = \sum_{i=1}^{n} \mu_i \qquad V[y] = \sum_{i=1}^{n} \sigma_i^2$$

Measurement errors are often the sum of many contributions, so frequently measured values can be treated as Gaussian r.v.s.

Central Limit Theorem (2)

The CLT can be proved using characteristic functions (Fourier transforms), see, e.g., SDA Chapter 10.

For finite n, the theorem is approximately valid to the extent that the fluctuation of the sum is not dominated by one (or few) terms.

Beware of measurement errors with non-Gaussian tails.

Good example: velocity component v_x of air molecules.

OK example: total deflection due to multiple Coulomb scattering. (Rare large angle deflections give non-Gaussian tail.)

Bad example: energy loss of charged particle traversing thin gas layer. (Rare collisions make up large fraction of energy loss, cf. Landau pdf.)

Multivariate Gaussian distribution

Multivariate Gaussian pdf for the vector $\vec{x} = (x_1, \ldots, x_n)$:

$$f(\vec{x}; \vec{\mu}, V) = \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T V^{-1}(\vec{x}-\vec{\mu})\right]$$

$\vec{x}$, $\vec{\mu}$ are column vectors, $\vec{x}^T$, $\vec{\mu}^T$ are transpose (row) vectors,

$$E[x_i] = \mu_i, \, , \qquad \mathrm{cov}[x_i, x_j] = V_{ij} \; .$$

For $n = 2$ this is

$$f(x_1, x_2, ; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

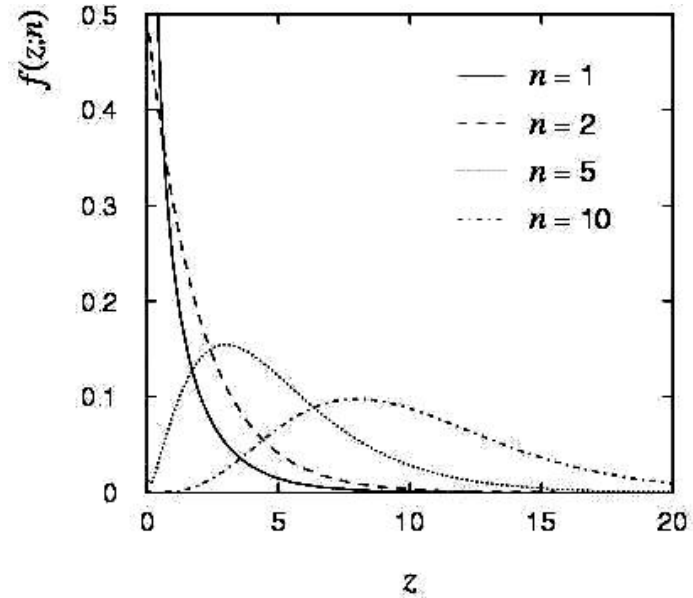
$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

where $\rho = \mathrm{cov}[x_1, x_2]/(\sigma_1\sigma_2)$ is the correlation coefficient.

Chi-square (χ^2) distribution

The chi-square pdf for the continuous r.v. z ($z \geq 0$) is defined by

$$f(z;n) = \frac{1}{2^{n/2}\Gamma(n/2)} z^{n/2-1} e^{-z/2}$$



$n = 1, 2, ... =$ number of 'degrees of freedom' (dof)

$$E[z] = n \,, \qquad V[z] = 2n \,.$$

For independent Gaussian x_i, $i = 1, ..., n$, means μ_i, variances σ_i^2,

$$z = \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}$$ follows χ^2 pdf with n dof.

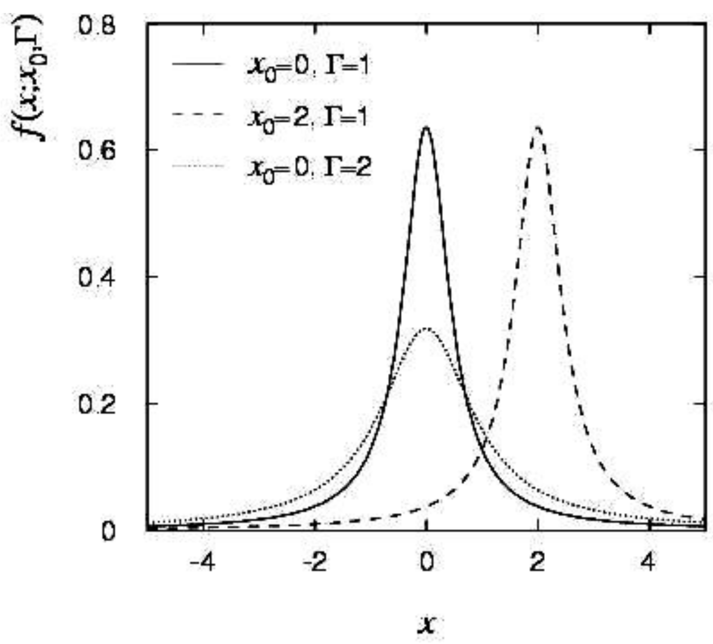
Example: goodness-of-fit test variable especially in conjunction with method of least squares.

Cauchy (Breit-Wigner) distribution

The Breit-Wigner pdf for the continuous r.v. x is defined by

$$f(x;\Gamma,x_0) = \frac{1}{\pi}\frac{\Gamma/2}{\Gamma^2/4 + (x-x_0)^2}$$

($\Gamma = 2$, $x_0 = 0$ is the Cauchy pdf.)



$E[x]$ not well defined, $V[x] \rightarrow \infty$.

x_0 = mode (most probable value)

Γ = full width at half maximum

Example: mass of resonance particle, e.g. ρ, K^*, ϕ^0, ...

Γ = decay rate (inverse of mean lifetime)

Landau distribution

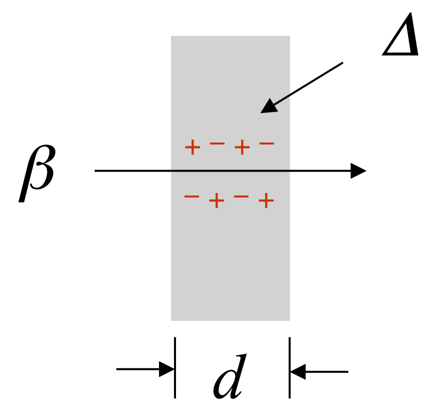
For a charged particle with $\beta = v/c$ traversing a layer of matter of thickness d, the energy loss Δ follows the Landau pdf:

$$f(\Delta;\beta) = \frac{1}{\xi}\phi(\lambda) \;,$$

$$\phi(\lambda) = \frac{1}{\pi}\int_0^\infty \exp(-u\ln u - \lambda u)\sin \pi u \, du \;,$$

$$\lambda = \frac{1}{\xi}\left[\Delta - \xi\left(\ln\frac{\xi}{\epsilon'} + 1 - \gamma_{\mathrm{E}}\right)\right] \;,$$

$$\xi = \frac{2\pi N_{\mathrm{A}} e^4 z^2 \rho \sum Z}{m_{\mathrm{e}} c^2 \sum A}\frac{d}{\beta^2} \;, \qquad \epsilon' = \frac{I^2 \exp \beta^2}{2 m_{\mathrm{e}} c^2 \beta^2 \gamma^2} \;.$$



L. Landau, J. Phys. USSR **8** (1944) 201; see also
W. Allison and J. Cobb, Ann. Rev. Nucl. Part. Sci. **30** (1980) 253.

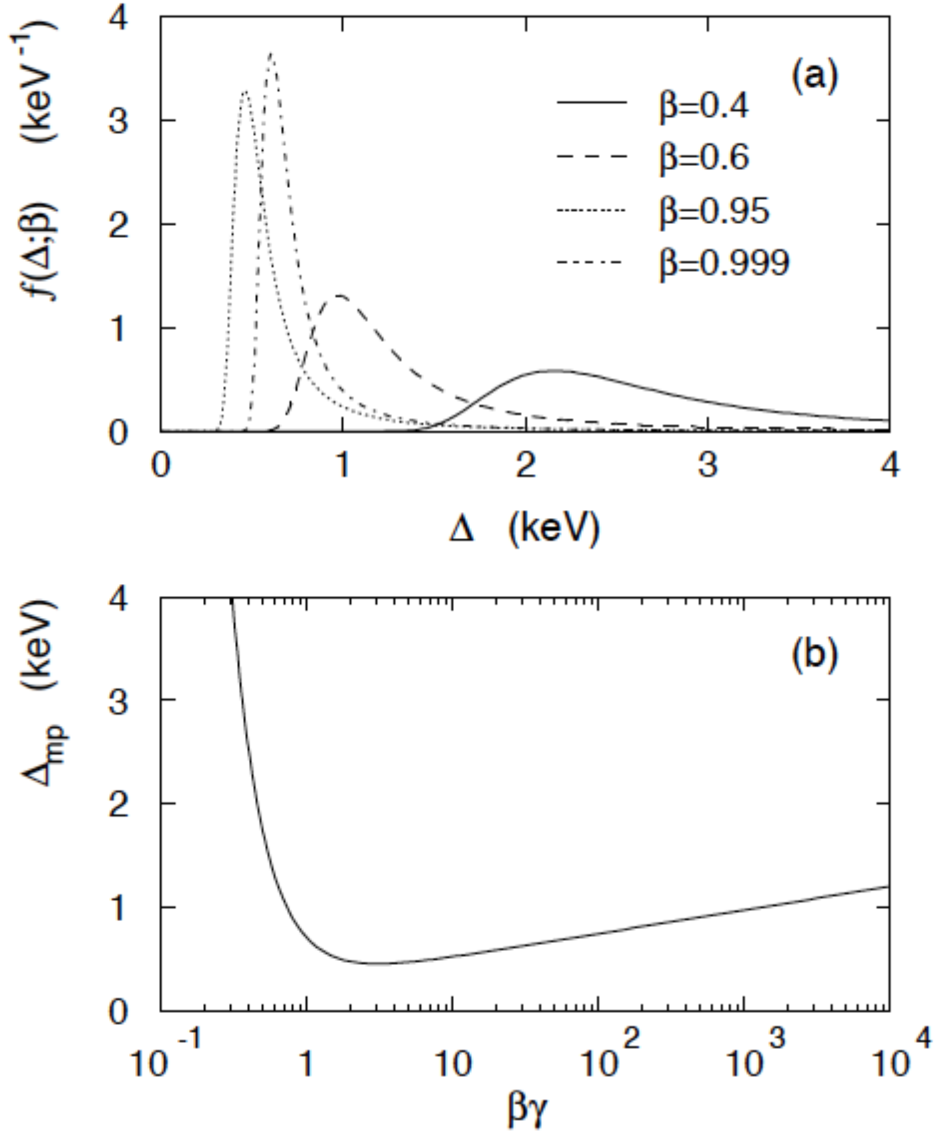
Landau distribution (2)

Long 'Landau tail'

→ all moments ∞

Mode (most probable value) sensitive to β ,

→ particle i.d.



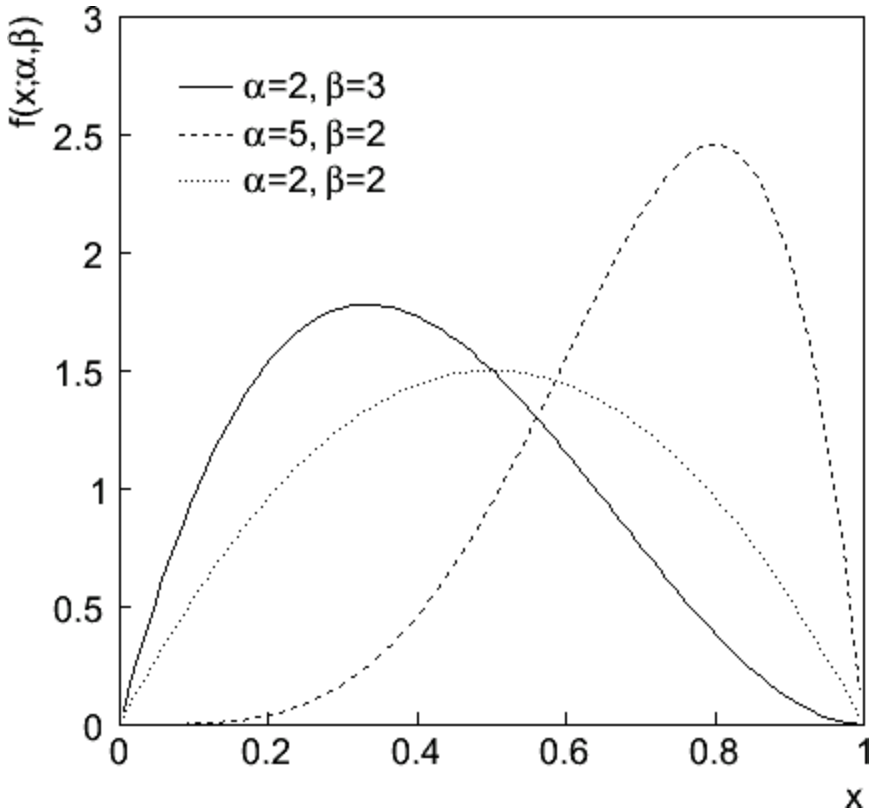
Beta distribution

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}$$

$$E[x] = \frac{\alpha}{\alpha+\beta}$$

$$V[x] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Often used to represent pdf of continuous r.v. nonzero only between finite limits.



Gamma distribution

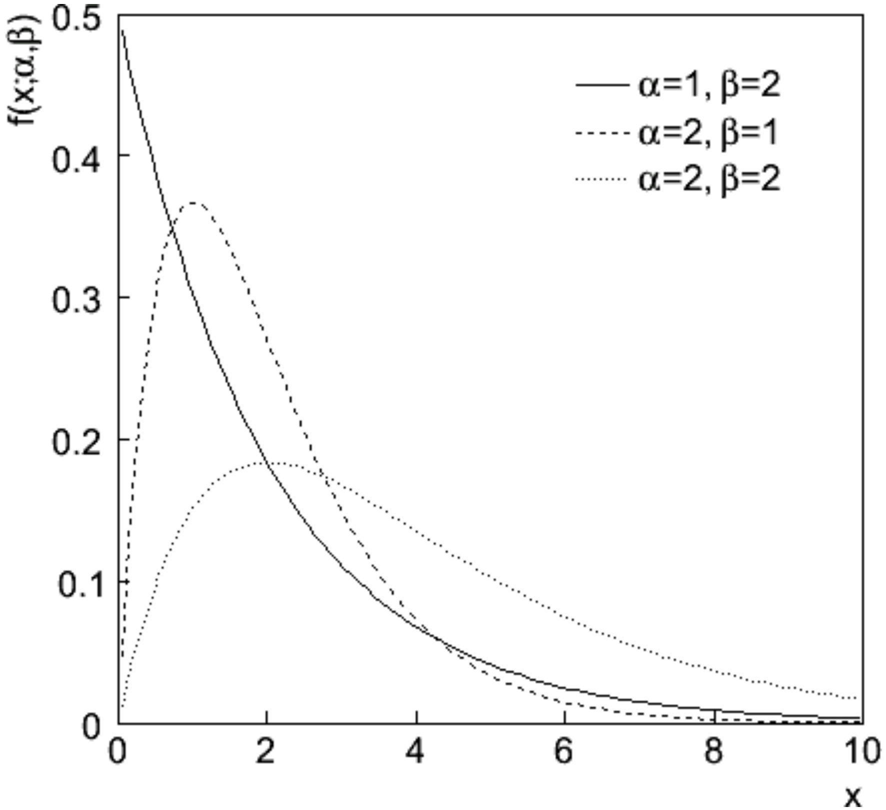
$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, x^{\alpha-1}\, e^{-x/\beta}$$

$$E[x] = \alpha\beta$$

$$V[x] = \alpha\beta^2$$

Often used to represent pdf of continuous r.v. nonzero only in [0,∞].

Also e.g. sum of *n* exponential r.v.s or time until *n*th event in Poisson process ~ Gamma



Student's *t* distribution

$$f(x;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}$$

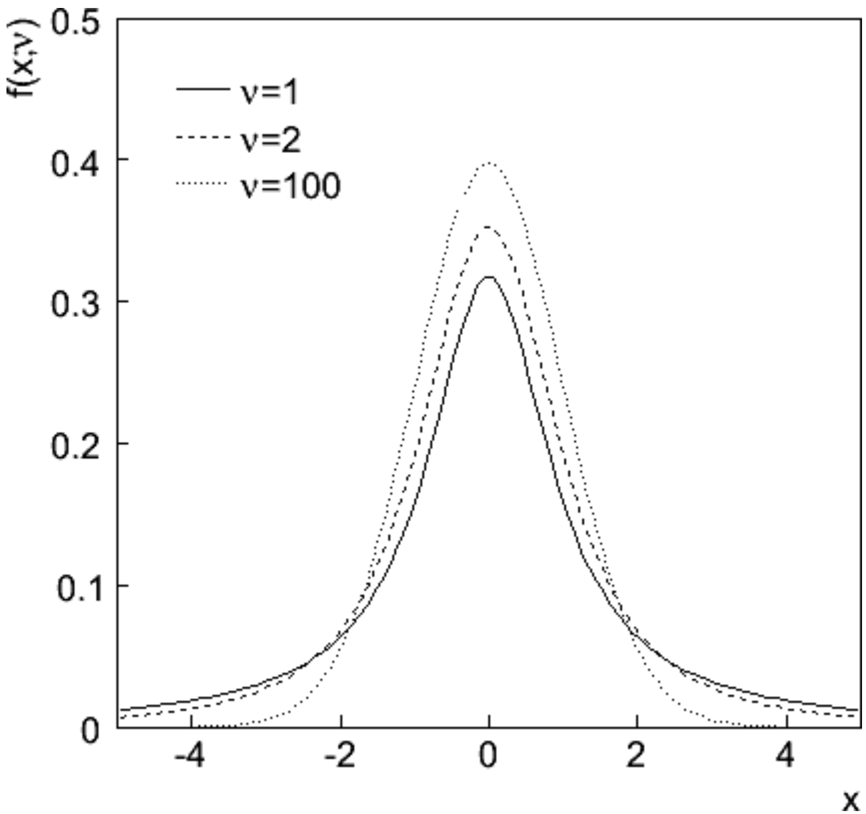
$$E[x] = 0 \quad (\nu > 1)$$

$$V[x] = \frac{\nu}{\nu - 2} \quad (\nu > 2)$$

ν = number of degrees of freedom
(not necessarily integer)

ν = 1 gives Cauchy,

$\nu \rightarrow \infty$ gives Gaussian.



Student's t distribution (2)

If $x \sim$ Gaussian with $\mu = 0$, $\sigma^2 = 1$, and

$z \sim \chi^2$ with n degrees of freedom, then

$t = x \,/\, (z/n)^{1/2}$ follows Student's t with $\nu = n$.

This arises in problems where one forms the ratio of a sample mean to the sample standard deviation of Gaussian r.v.s.

The Student's t provides a bell-shaped pdf with adjustable tails, ranging from those of a Gaussian, which fall off very quickly, ($\nu \to \infty$, but in fact already very Gauss-like for $\nu =$ two dozen), to the very long-tailed Cauchy ($\nu = 1$).

Developed in 1908 by William Gosset, who worked under the pseudonym "Student" for the Guinness Brewery.

Characteristic functions

The characteristic function $\varphi_x(k)$ of an r.v. x is defined as the expectation value of e^{ikx} (~Fourier transform of x):

$$\phi_x(k) = E[e^{ikx}] = \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$

Useful for finding moments and deriving properties of sums of r.v.s.

For well-behaved cases (true in practice), characteristic function is equivalent to pdf and vice versa, i.e., given one you can in principle find the other (like Fourier transform pairs).

Characteristic functions: examples

Distribution	p.d.f.	$\phi(k)$
Binomial	$f(n;N,p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$	$[p(e^{ik}-1)+1]^N$
Poisson	$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu}$	$\exp[\nu(e^{ik}-1)]$
Uniform	$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{i\beta k} - e^{i\alpha k}}{(\beta-\alpha)ik}$
Exponential	$f(x;\xi) = \frac{1}{\xi} e^{-x/\xi}$	$\frac{1}{1-ik\xi}$

Characteristic functions: more examples

Distribution	p.d.f.	$\phi(k)$
Gaussian	$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$	$\exp(i\mu k - \frac{1}{2}\sigma^2 k^2)$
Chi-square	$f(z;n) = \frac{1}{2^{n/2}\Gamma(n/2)} z^{n/2-1} e^{-z/2}$	$(1-2ik)^{-n/2}$
Cauchy	$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$	$e^{-\|k\|}$

Moments from characteristic function

Suppose we have a characteristic function $\varphi_z(k)$ of a variable z.

By differentiating m times and evaluating at $k = 0$ we find:

$$\begin{aligned} \frac{d^m}{dk^m}\phi_z(k)\bigg|_{k=0} &= \frac{d^m}{dk^m}\int e^{ikz} f(z)dz\bigg|_{k=0} \\ &= i^m \int z^m f(z)dz \\ &= i^m \mu'_m \end{aligned}$$

where $\mu_m' = E[x^m]$ is the m^{th} algebraic moment of z.

So if we have the characteristic function we can find the moments of an r.v. even if we don't have an explicit formula for its pdf.

Example of moments from characteristic function

For example, using the characteristic function of a Gaussian

$$\phi_x(k) = \exp\left(i\mu k - \tfrac{1}{2}\sigma^2 k^2\right)$$

we can find the mean and variance,

$$\begin{aligned}
E[x] &= -i\frac{d}{dk}[\exp(i\mu k - \tfrac{1}{2}\sigma^2 k^2)]\Big|_{k=0} = \mu, \\
V[x] &= E[x^2] - (E[x])^2 \\
&= -\frac{d^2}{dk^2}[\exp(i\mu k - \tfrac{1}{2}\sigma^2 k^2)]\Big|_{k=0} - \mu^2 = \sigma^2.
\end{aligned}$$

Limiting cases of distributions from c.f.

Characteristic function of the binomial distribution is

$$\phi(k) = [p(e^{ik} - 1) + 1]^N$$

Taking limit $p \to 0$, $N \to \infty$, with $\nu = pN$ constant gives

$$\phi(k) = \left(\frac{\nu}{N}(e^{ik} - 1) + 1\right)^N \to \exp[\nu(e^{ik} - 1)]$$

which is the characteristic function of the Poisson distribution.

In a similar way one can show that the Poisson distribution with mean ν becomes a Gaussian with mean ν and standard deviation $\sqrt{\nu}$ in the limit $\nu \to \infty$.

Addition theorem for characteristic functions

Suppose we have n independent random variables $x_1, \ldots, x_n$ with pdfs $f_1(x_1), \ldots, f_n(x_n)$ and characteristic functions $\varphi_1(k), \ldots, \varphi_n(k)$.

Consider the sum: $z = \sum_{i=1}^{n} x_i$ Its characteristic function is

$$
\begin{aligned}
\phi_z(k) &= \int \ldots \int \exp\left(ik \sum_{i=1}^{n} x_i\right) f_1(x_1) \ldots f_n(x_n) dx_1 \ldots dx_n \\
&= \int e^{ikx_1} f_1(x_1) dx_1 \ldots \int e^{ikx_n} f_n(x_n) dx_n \\
&= \phi_1(k) \ldots \phi_n(k).
\end{aligned}
$$

So the characteristic function of a sum is the product of the individual characteristic functions.

Addition theorem, continued

The pdf of the sum z can be found from the inverse (Fourier) transform:

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_z(k)\, e^{-ikz}\, dk$$

Can e.g. show that for n independent $x_i \sim \text{Gauss}(\mu_i, \sigma_i)$, the sum

$$z = \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}$$

follows a chi-square distribution for n degrees of freedom.

Also can be used to prove Central Limit Theorem and solve many other problems involving sums of random variables (SDA Ch. 10).