2. The outcome of a measurement consists of two independent random values, x and y, that follow the pdfs

$$
\begin{aligned}
f(x|\theta_1, \theta_2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\theta_1-\theta_2)^2/2\sigma^2} \;, \\
g(y|\theta_2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\theta_2)^2/2\sigma^2} \;.
\end{aligned}
$$

Consider the standard deviation σ (same for x and y) to be known.

(a) Write down the log-likelihood function for θ_1 and θ_2.

[4]

(b) Show that the Maximum-Likelihood estimators for θ_1 and θ_2 are

$$
\begin{aligned}
\hat{\theta}_1 &= x - y \;, \\
\hat{\theta}_2 &= y \;.
\end{aligned}
$$

[6]

(c) Show that the estimators given above are unbiased. Find their exact variances, their covariance and correlation coefficient.

[10]

(d) Show with the help of a sketch how the standard deviations of $\hat{\theta}_1$ and $\hat{\theta}_2$ can be determined from a contour of the log-likelihood function. Label all of the relevant features.

[6]

(e) Suppose θ_1 is the parameter of interest and we regard θ_2 as a nuisance parameter. Find the profiled value $\hat{\hat{\theta}}_2(\theta_1)$ and using this show that the log of the profile likelihood for θ_1 can be written

$$
\ln L_{\mathrm{p}}(\theta_1) = -\frac{1}{4}\frac{(x-y-\theta_1)^2}{\sigma^2} + C
$$

where C represents terms that do not depend on the unknown parameters.

[8]

Show that the variance of $\hat{\theta}_1$ as determined directly from the second derivative of the profile likelihood is the same as found in (c).

[6]