2(a) [4 marks] The variables x and y are independent, so the likelihood function is given by the product of the two pdfs, i.e.,

$$L(\theta_1, \theta_2) = f(x|\theta_1, \theta_2) g(y|\theta_2) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x - \theta_1 - \theta_2)^2}{2\sigma^2} - \frac{(y - \theta_2)^2}{2\sigma^2} \right] .$$

The log-likelihood function is therefore (4 marks)

$$\ln L(\theta_1, \theta_2) = -\frac{1}{2}\frac{(x - \theta_1 - \theta_2)^2}{\sigma^2} - \frac{1}{2}\frac{(y - \theta_2)^2}{\sigma^2} + C ,$$

where $C = -\ln 2\pi\sigma^2$ is a constant (i.e., does not depend on θ_1 or θ_2) and thus can be dropped.

2(b) [6 marks] To find the ML estimators we set the derivatives of $\ln L$ with respect to the parameters equal to zero:

$$\frac{\partial \ln L}{\partial \theta_1} = -\frac{1}{2}\frac{2(x - \theta_1 - \theta_2)(-1)}{\sigma^2} = 0 , \tag{4}$$

$$\frac{\partial \ln L}{\partial \theta_2} = -\frac{1}{2}\frac{2(x - \theta_1 - \theta_2)(-1)}{\sigma^2} - \frac{1}{2}\frac{(y - \theta_2)(-1)}{\sigma^2} = 0 . \tag{5}$$

From Eq. (5) we get $\theta_1 + \theta_2 = x$, and from this the first term in Eq. (5) is zero. We therefore find the ML estimators (6 marks)

$$\hat{\theta}_1 = x - y ,$$

$$\hat{\theta}_2 = y .$$

2(c) [10 marks] From the pdfs of x and y given we can see the expectation values and variances are

$$E[x] = \theta_1 + \theta_2 ,$$

$$E[y] = \theta_2 ,$$

$$V[x] = V[y] = \sigma^2 .$$

The expectation values of $\hat{\theta}_1$ and $\hat{\theta}_2$ are (3 marks)

$$E[\hat{\theta}_1] = E[x - y] = E[x] - E[y] = \theta_1 + \theta_2 - \theta_2 = \theta_1 ,$$

$$E[\hat{\theta}_2] = E[y] = \theta_2 ,$$

and therefore we see that both estimators are unbiased. The variances of $\hat{\theta}_1$ and $\hat{\theta}_2$ are (3 marks)

$$V[\hat{\theta}_1] = V[x-y] = V[x] + V[y] = 2\sigma^2 \ ,$$
$$V[\hat{\theta}_2] = V[y] = \sigma^2 \ ,$$

and the covariance of $\hat{\theta}_1$ and $\hat{\theta}_2$ is (2 marks)

$$\text{cov}[\hat{\theta}_1, \hat{\theta}_2] = \text{cov}[x-y, y] = \text{cov}[x, y] - \text{cov}[y, y] = 0 - V[y] = -\sigma^2 \ .$$

Combining the ingredients above gives the correlation coefficient (2 marks)

$$\rho = \frac{\text{cov}[\hat{\theta}_1, \hat{\theta}_2]}{\sqrt{V[\hat{\theta}_1]V[\hat{\theta}_2]}} = \frac{-\sigma^2}{\sqrt{\sigma^2 \times 2\sigma^2}} = -\frac{1}{\sqrt{2}} \ .$$

2(d) [6 marks] Figure 2 shows a contour of the log-likelihood $\ln L(\theta, \theta_2) = \ln L_{\text{max}} - 1/2$, which is centred about the ML estimators $(\hat{\theta}_1, \hat{\theta}_2)$ (2 marks). The standard deviations are determined from the distance from the ML estimators to the tangent lines to the contour (3 marks). (In the large sample limit the contour is symmetric so the distance to either tangent line can be used.) The negative correlation is indicated by the tilt of the contour from upper left to lower right (1 mark).

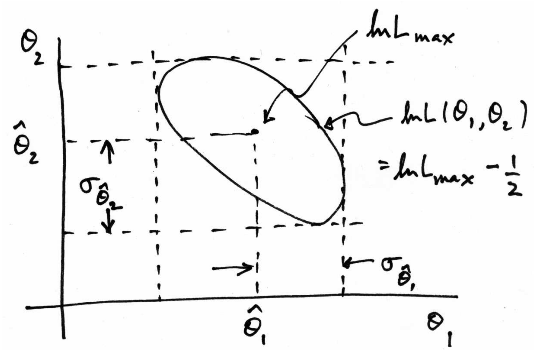

Figure 2: The standard deviations $\sigma_{\hat{\theta}_1}$ and $\sigma_{\hat{\theta}_2}$ are determined from the tangent lines to the contour of $\ln L(\theta_1, \theta_2) = \ln L_{\text{max}} - 1/2$.

2(e) [14 marks] From Eq. (5) above setting the derivative of $\ln L$ with respect to θ_2 equal to zero we have

$$\frac{x - \theta - \theta_2}{\sigma^2} + \frac{y - \theta_2}{\sigma^2} = 0 \ .$$

Treating θ_1 as fixed and solving for θ_2 gives the profiled value (4 marks)

$$\hat{\hat{\theta}}_2(\theta_1) = \frac{x + y - \theta_1}{2} \ .$$

The profile likelihood $L_{\text{p}}(\theta_1)$ is defined by evaluating $L(\theta_1, \theta_2)$ with $\hat{\hat{\theta}}_2(\theta_1)$ as found above. After dropping constant terms we find (4 marks)

$$
\begin{aligned}
\ln L_{\mathrm{p}}(\theta_1) &= -\frac{1}{2\sigma^2}\left[\left(x-\theta_1-\frac{x+y-\theta_1}{2}\right)^2+\left(y-\frac{x+y-\theta_1}{2}\right)^2\right] \\
&= -\frac{1}{2\sigma^2}\left[\left(\frac{x}{2}-\frac{y}{2}-\frac{\theta_1}{2}\right)^2+\left(\frac{y}{2}-\frac{x}{2}+\frac{\theta_1}{2}\right)^2\right] \\
&= -\frac{1}{4}\frac{x-y-\theta_1)^2}{\sigma^2}\ .
\end{aligned}
$$

The derivatives of $\ln L_{\mathrm{p}}$ are (3 marks)

$$
\begin{aligned}
\frac{\partial \ln L_{\mathrm{p}}}{\partial \theta_1} &= \frac{1}{2}\frac{x-y-\theta_1}{\sigma^2}\ , \\
\frac{\partial^2 \ln L_{\mathrm{p}}}{\partial \theta_1^2} &= -\frac{1}{2\sigma^2}\ .
\end{aligned}
$$

Using the profile likelihood to find the Fisher information therefore gives (2 marks)

$$
I(\theta) = -E\left[\frac{\partial^2 \ln L_{\mathrm{p}}}{\partial \theta_1^2}\right] = \frac{1}{2\sigma^2}
$$

Using this to determine (approximately) the variance thus gives the same as the exact result found above (1 mark)

$$
V[\hat{\theta}_1] \approx I^{-1}(\theta) = 2\sigma^2\ .
$$