PH2910 - Discussion Session - Week 1

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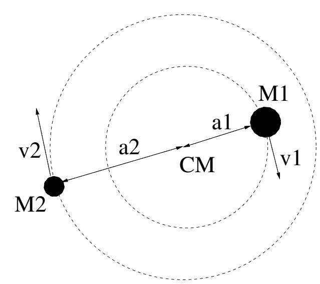
Exercise: Consider a spectroscopic binary system, i.e. one in which the presence of two stars is deduced from the separation of spectral lines due to the Doppler effect, into sets of double lines. The period of revolution, P_{orb}, as well as the radial velocities $v_{0,1}$ and $v_{0,2}$, of the stars along the line of sight, can be determined from the periodic changes oibserved in the spectrum. Consider further that the stars have circular orbits and that the inclination of the orbital plane with respect to the line of sight is i. Find the masses of the two stars, M_1 and M_2, as a function of the observables and of $\sin i$.

Solution:



The ratio of the distances between each star and the centre of mass, a_1 and a_2 (see figure), depends the star masses. From the definition of the centre of mass:

$$\begin{aligned} M_1 a_1 &= M_2 a_2 \\ M_1 a_1 &= M_2(a - a_1) \\ a_1(M_1 + M_2) &= a M_2 \end{aligned}$$

where $M = M_1 + M_2$ is the total mass and $a = a_1 + a_2$ is the separation between the stars. The next two results are immediate:

$$a_1 = \frac{aM_2}{M} \quad (1)$$

$$a_2 = \frac{aM_1}{M} \quad (2)$$

$$\frac{a_1}{a_2} = \frac{M_2}{M_1} \quad (3)$$

On the other hand, for circular orbits, the star velocities in the plane of the orbits, v_1 and v_2, are constant and related to the period of revolution, P_{orb} and the angular velocity, ω, by:

$$\omega = \frac{2\pi}{P_{orb}}$$

$$a_1\omega = a_1\frac{2\pi}{P_{orb}}$$

note that ω is constant and equal for both stars and that $a_1\omega = v_1$. This leads immediately to:

$$\frac{a_1}{a_2} = \frac{v_1}{v_2} = \frac{v_{0,1}\sin i}{v_{0,2}\sin i} = \frac{v_{0,1}}{v_{0,2}} \quad (4)$$

and, from equations 3 and 4:

$$\frac{v_1}{v_2} = \frac{M_2}{M_1} \quad (5)$$

Now, the centripetal force, F_c, which keeps the stars in their orbits, can be written $M_{star}\omega^2 distance_{star-CM}$. We also know that the force which is applied on each star is given by Newton's law of gravity. Putting the two together we get:

$$F_c = M_1\omega^2 a_1 = \frac{GM_1M_2}{a}$$

$$F_c = M_2\omega^2 a_2 = \frac{GM_1M_2}{a}$$

i.e.

$$\begin{aligned}\omega^2 a_1 &= \frac{GM_1}{a^2}\\ \omega^2 a_2 &= \frac{GM_2}{a^2}\end{aligned}$$

and summing the two equations:

$$\begin{aligned}\omega^2(a_1+a_2) &= \frac{G}{a^2}(M_1+M_2)\\ \omega^2(a_1+a_2)^3 &= G(M_1+M_2)\\ \omega^2(\frac{v_1}{\omega}+\frac{v_2}{\omega})^3 &= G(M_1+M_2)\\ \frac{(v_1+v_2)^3}{\omega} &= G(M_1+M_2)\end{aligned}$$

but $v_1 = v_{0,1}/\sin i$ (see figure 1) and $v_1 = v_{0,1}/\sin i$ and $\omega = 2\pi/P_{orb}$ so we get:

$$M_1 + M_2 = \frac{P_{orb}}{2\pi G}\frac{(v_{0,1}+v_{0,2})^3}{\sin i^3} \quad (6)$$

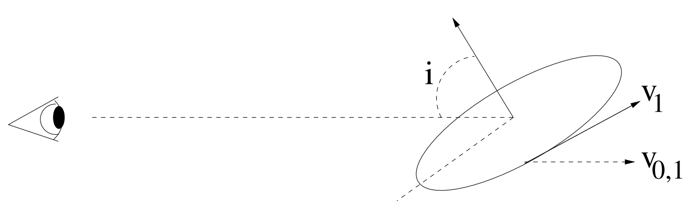


Figure 1: Inclination angle and radial velocity.

Combining equations 4 and 6 we get:

$$\begin{aligned}M_1(1+\frac{v_{0,1}}{v_{0,2}}) &= \frac{P_{orb}}{2\pi G}\frac{(v_{0,1}+v_{0,2})^3}{\sin i^3}\\ M_1(\frac{v_{0,1}+v_{0,2}}{v_{0,2}}) &= \frac{P_{orb}}{2\pi G}\frac{(v_{0,1}+v_{0,2})^3}{\sin i^3}\end{aligned}$$

i.e.

$$M_1 = \frac{v_{0,2} P_{orb}}{2\pi G} \frac{(v_{0,1} + v_{0,2})^2}{\sin i^3}$$

In the same way, the mass of the second star is given by:

$$M_2 = \frac{v_{0,1} P_{orb}}{2\pi G} \frac{(v_{0,1} + v_{0,2})^2}{\sin i^3}$$

Masses of stars in spectroscopic binaries can be estimated from these formulae. If the binary system is also eclipsing (i.e. one star passes periodically in front of the other and produces a characteristic variation of the system's brightness), reliable values for the inclination angle may be found. Circular orbits can be inferred if the position of spectral lines (and therefore the radial velocity) varies approximately with the sine of time (why?).