# PH2910 - Discussion Session - Week 2

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23 January 2006

Exercise 1: The sun is $4.6 \times 10^9$ years old. Its luminosity is $L_\odot = 3.9 \times 10^{26} W$ and its mass is $M_\odot = 2 \times 10^{30} kg$.

a) How much mass is converted into energy in the sun in one second?

b) How much mass was converted into energy by the sun assuming a constant value of the luminosity?

c) What fraction of the sun's mass was converted into energy so far?

d) The measured density of particles in the solar wind at $1AU$ from the sun is around $10^7 m^{-1}$, and their average velocity with respect to the sun is $470\ km\ s^{-1}$. What fraction of the sun's mass is lost per second by solar wind emission? Assume that the solar wind is constant and is emitted isotropically by the sun, and that it is composed only of protons and electrons in equal numbers. $1AU = 1.4960 \times 10^{11} m$

Solution:

a) Energy is produced in the sun through nuclear reactions. The amount of energy produced in a reaction between two nuclei corresponds to the mass lost in the reaction given by $E = mc^2$. The energy produced in one second, $E_{1s}$, corresponds to ($W = Js^{-1}$):

$$\begin{aligned} E_{1s} &= 3.9 \times 10^{26}\ [J] = mc^2 \\ m &= \frac{3.9 \times 10^{26}\ [J]}{(3 \times 10^8)^2\ [m^2 s^{-2}]} \\ m &= 4.3 \times 10^9\ [Nms^{-2}] = 4.3 \times 10^9\ [kg] \end{aligned}$$

b) This is a simple multiplication of the mass transformed into energy per second times the age of the sun in seconds:

$$\begin{aligned} m_{tot} &= 4.3 \times 10^9\ [kg] \cdot 4.6 \times 10^9\ [year] \cdot 365\ [days/year] \cdot 24\ [hr/day] \cdot 3600\ [s/hr] \\ &= 4.3 \times 10^9 \cdot 1.5 \times 10^{17}\ [s] = 6.5 \times 10^{26}\ [kg] \end{aligned}$$

c) Too simple for words...

$$m_{tot}/M_\odot = \frac{6.5 \times 10^{26}\ [kg]}{2 \times 10^{30}\ [kg]} = 0.00032 = 0.032\%$$

i.e. a very small fraction of the sun's mass has been converted into energy in 4.6 billion years. We can normally consider the mass of stars to remain constant troughout their life.

d) With *very* rough simplifications: considering the solar wind to be composed only of protons and electrons in equal proportions, the total mass corresponding to the particles which cross a cross section surface at 1 AU from the sun is:

$$\begin{aligned} F_w &= 10^7\ [part/m^3]\ (0.5m_p + 0.5m_e)\ 470km.s^{-1} \\ &= 10^7\ [m^{-3}]\ (0.5 \cdot 1.672 \times 10^{-27} + 0.5 \cdot 9.109 \times 10^{-31})\ [kg]\ 4.7 \times 10^5\ [m\ s^{-1}] \\ &= 3.93 \times 10^{-15}\ [kg\ m^{-2} s^{-1}] \end{aligned}$$

considering the solar wind to be constant and isotropic, the total mass emitted per second by the sun is the mass of particles which cross a sphere of radius 1AU centered on the sun:

$$\begin{aligned} m_w &= \int_0^{4\pi} F_w d\Omega = F_w\ 4\pi\ R^2 = F_w\ 4\pi\ (1AU)^2 \\ &= 3.93 \times 10^{-15}\ [kg\ m^{-2}.s^{-1}]\ 4\pi\ (1.496 \times 10^{11})^2\ [m^2] \\ &= 1.106 \times 10^9\ [kg\ s^{-1}] \end{aligned}$$

Comparing $m_w$ with the solution from a) we see that they are of roughly the same order of magnitude. The solar mass can still be considered constant.

Exercise 2: Find a lower bound to the pressure at the centre of the sun using the equation of motion for stars in hydrostatic equilibrium.

Solution: Integrating the equation of motion:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

integrating over the whole star (i.e. from $m = 0$ to $m = M_\odot$):

$$\int_0^{M_\odot} dP = -\int_0^{M_\odot} \frac{Gm}{4\pi r^4} dm$$
$$P(M_\odot) - P(0) = -\frac{G}{4\pi} \int_0^{M_\odot} \frac{m}{r^4} dm$$

But $P(M_\odot) = 0$ (it is the pressure at the surface of the sun) and so:

$$P(0) = \frac{G}{4\pi} \int_0^{M_\odot} \frac{m}{r^4} dm$$

To do this integral we should know the density profile of the star, $\rho(r)$, to know the transformation between $m$ and $r$. Instead, we can determine a lower bound on the pressure at the centre of the sun, $P(0)$, by replacing $r$ by the solar radius, $R_\odot$, so we get the inequality:

$$P(0) > \frac{G}{4\pi} \int_0^{M_\odot} \frac{m}{R_\odot^4} dm$$
$$P(0) > \frac{G}{4\pi} \frac{M_\odot^2}{R_\odot^4}$$

Using the values $M_\odot = 2 \times 10^{30}$ $[kg]$, $G = 6.7 \times 10^{-11}$ $[m^3 kg^{-1} s^{-2}]$ and $R_\odot = 7 \times 10^8$ $[m]$ we get:

$$P(0) > 4.4 \times 10^{13} \; [kg \; m \; s^{-2}.m^{-2}] = 4.4 \times 10^{13} \; [N \; m^{-2} \equiv Pa]$$
$$P(0) > \frac{4.4 \times 10^{13} \; [Pa]}{101325 \; [Pa/atm]} = 434 \times 10^6 atm$$

i.e. the pressure at the centre of the sun exceeds around 430 million atmospheres.

Exercise 3: Determine the average temperature, $\overline{T}$, in the sun using the approximate expression for the total gravitational potential energy:

$$\Omega = -\alpha \frac{GM_\odot^2}{R_\odot}$$

where $\alpha$ is a constant of order 1 which depends on the distribution of matter in the sun, and the result obtained from the virial theorem assuming an ideal gas:

$$\frac{3k\overline{T}M}{2m_g} = -\frac{1}{2}\Omega$$

Use the following numbers:

- $M_\odot = 2 \times 10^{30}$ $[kg]$
- $R_\odot = 7 \times 10^{8}$ $[m]$
- $m_g = 1.7 \times 10^{-27}$ $[kg]$ (i.e. hydrogen)
- $k = 1.4 \times 10^{-23}$ $[J\ K^{-1}]$ (Boltzmann's constant)
- $k = 6.7 \times 10^{-11}$ $[m^3 kg^{-1} s^{-2}]$
- assume $\alpha = 0.5$

Solution:

$$\begin{aligned}
\frac{3k\overline{T}M}{2m_g} &= -\frac{1}{2}\Omega = \frac{\alpha}{2}\frac{GM^2}{R} \\
\overline{T} &= \frac{\alpha m_g GM}{3kR} \\
\overline{T} &= 3.8 \times 10^6\ [kg\ m^2\ s^{-2} J^{-1} K] = 3.8 \times 10^6\ [N\ m\ N^{-1} m^{-1} K] \\
\overline{T} &= 3.8 \times 10^6\ [K]
\end{aligned}$$

i.e. the average temperature in the sun is around 3.8 million degrees Kelvin.