PH2910 - Discussion Session - Week 5

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Exercise 1: (this is exercise 3.2 in *Prialnik*)

Assuming that the gas pressure, P_{gas}, is a fraction β of the total pressure, P, which is constant throughout the star, and defining the total internal energy of the star, U, as

$$U = \int_0^M (u_{gas} + u_{rad})dm, \tag{1}$$

where u_{gas} and u_{rad} are the specific energies corresponding to the gas and the radiation. Show that the virial theorem in the form

$$\Omega = -3\int_0^M \frac{P}{\rho}dm, \tag{2}$$

leads to the following expressions for the total energy of the star, E:

$$E = \frac{\beta}{2}\Omega \tag{3}$$

$$E = \frac{-\beta}{2-\beta}U \tag{4}$$

for a classical (i.e. non-relativistic), non-degenerate gas. Note in particular the limiting cases $\beta \to 0$ and $\beta \to 1$.

Solution:

We would like to express the total energy, $E = U + \Omega$ [1], in terms of the internal energy, U, or of the gravitational energy, Ω, but not both. So we

[1] This assumes that the star is in hydrostatic equilibrium, i.e. there is no macroscopic flow of matter, which would mean we need an extra term in the expression of E, corresponding to the macroscopic kinetic energy.

need a relation between U and Ω, which is given by the virial theorem 2. As equation 2 in expressed in terms of P/ρ, we need to express U in terms of the pressure and the density. So we need the expressions for u_{gas} and u_{rad} in terms of pressure and density, so we can use formula 1 to find U as function of P and ρ.

From $P_{gas} = \beta P$, the radiation pressure, P_{rad}, is given by $P_{rad} = (1-\beta)P$, with β constant throughout the star.

In an ideal gas with a number density (number of particles per unit volume) n, at a temperature T, the internal energy density is given by $\frac{3}{2}nkT$. The pressure is given by $P_{gas} = nkT$. Therefore, the gas specific internal energy (energy density per unit mass), u_{gas}, is given by

$$u_{gas} = \frac{3}{2}\frac{P_{gas}}{\rho} = \frac{3}{2}\frac{\beta P}{\rho}$$

where we divided the internal energy per unit volume by the mass per unit volume, ρ, to get the specific internal energy.

For radiation we have [2]:

$$u_{rad} = \frac{3P_{rad}}{\rho} = \frac{3(1-\beta)P}{\rho}$$

This results from assuming a photon gas in thermodynamical equilibrium, which corresponds to a number density of photons, $n(\nu)$, with frequencies in the interval $(\nu, \nu + d\nu)$ given by Planck's blackbody distribution:

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{d\nu}{e^{\frac{h\nu}{kT}} - 1}$$

This leads to a radiation pressure (exerted on the gas particles) $\frac{1}{3}aT^4$, where a is a constant. The blackbody distribution also results in a radiation energy density given by aT^4, and therefore to a radiation specific energy given by $u_{rad} = aT^4/\rho = 3P_{rad}/\rho$.

Summing u_{gas} and u_{rad} we get:

$$u_{gas} + u_{rad} = \frac{3}{2}\frac{\beta P}{\rho} + \frac{3(1-\beta)P}{\rho}$$

$$u_{gas} + u_{rad} = \frac{3}{2}(2-\beta)\frac{P}{\rho}$$

$$\frac{P}{\rho} = \frac{u_{gas} + u_{rad}}{\frac{3}{2}(2-\beta)}$$

[2] *Prialnik*, section 3.5.

Using the virial theorem 2, we get:

$$\Omega = -3\frac{3}{2}\int_0^M \frac{u_{gas} + u_{rad}}{(2-\beta)} dm$$

As β is constant throughout the star, we can simplify this expression:

$$\Omega = -\frac{2}{2-\beta}\int_0^M (u_{gas} + u_{rad}) dm$$

which, from 1, is simply:

$$\Omega = -\frac{2}{2-\beta} U \tag{5}$$

Finally, E can be expressed as:

$$E = \Omega + U = -\frac{2}{2-\beta}U + U = \frac{-\beta}{2-\beta}U$$

proving equation 4. Inverting 5 we get:

$$E = \Omega + U = \Omega + \frac{-(2-\beta)}{2}\Omega = \frac{\beta}{2}\Omega$$

which proves equation 3.

The limiting cases are $\beta \to 0$ and $\beta \to 1$. As β tends to 0, the pressure becomes increasingly dominated by the radiation pressure. In the limit of $\beta \sim 0$, from equation 3, the total energy becomes close to zero. This means that the system becomes unbound and the star "evaporates"[3]. In the other limiting case, the pressure is dominated by the gas pressure and the radiation pressure becomes negligible as β approaches 1. In this case, from equation 4, we obtainthe well known relations (see *Prialnik*, section 2.5):

$$E = \frac{\Omega}{2} = -U$$

[3]In a bound system, the total energy must be negative. In a star, this is achieved by the gravitational potential energy being larger in absolute value than the internal energy of the gas

Nuclear Fuel	Process	$T_{threshold}$ 10^6 K	Products	Energy per Nucleon (MeV)
H	p-p	~ 4	He	6.55
H	CNO	15	He	6.25
He	3α	100	C, O	0.61
C	C+C	600	O, Ne, Na, Mg	0.54
O	O+O	1000	Mg, S, P, Si	~ 0.3
Si	Nuc.eq.	3000	Co, Fe, Ni	≤ 0.18

Table 1: Major nuclear burning processes.

Exercise 2: (this is exercise 4.2 in *Prialnik*)

Estimate the minimal stellar mass required for central ignition of the different nuclear fuels, according to the threshold temperatures of table 1. Assume:

a) a density profile given by $\rho = \rho_c(1 - \frac{r^2}{R^2})$, where ρ_c is the density at the centre of the star and R is the star radius (see figure 1);

b) solar composition;

c) non-degeneracy.

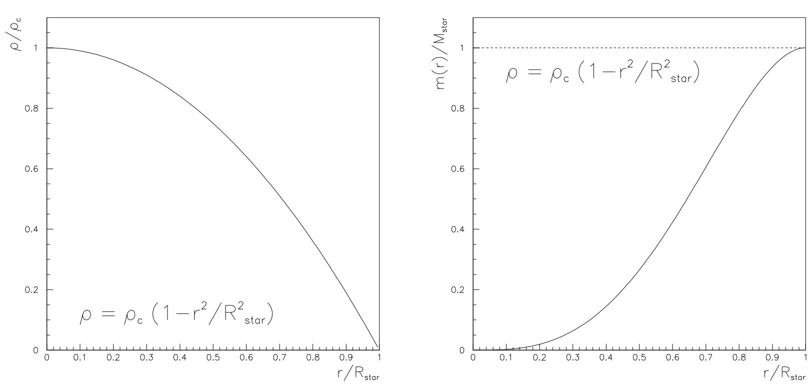

Figure 1: Density (left) and integrated mass (right) for a star with a density profile given by: $\rho(r) = \rho_c(1 - \frac{r^2}{R^2})$.

Solution:

The rate of nuclear burning is negligible at temperatured below the thresholds listed in the table and grows very rapidly with temperature above the threshold. We need to determine the temperature at the centre of the star, which is likely to be the highest in the star.

Assuming an ideal gas, the equation of state

$$P_{gas} = \frac{\mathcal{R}}{\mu}\rho T \tag{6}$$

relates temperature with density and pressure (see *Prialnik* section 3.3). $\mathcal{R}$ is the ideal gas constant ($\mathcal{R} = 8.3145 \times 10^5$ J kg^{-1} K^{-1}) and μ is the mean atomic weight:

$$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}$$

where μ_e and μ_I are the mean atomic weights for electrons and ions in the stellar material. These can be expressed as:

$$\frac{1}{\mu_I} = \sum_i \frac{X_i}{\mathcal{A}_i}$$

$$\frac{1}{\mu_e} = \sum_i \frac{X_i}{Z_i \mathcal{A}_i}$$

In the above expressions, the sum is over all ion species that are present in the star in mass fractions X_i and are characterized by atomic weights $\mathcal{A}_i$ and charge (or atomic number) Z_i. For the sum, we have $\mu_e \sim 1.17$ and $\mu \sim 0.61$ (see *Prialnik* section 3.3).

So we need to express P_{gas} and ρ as a function of the total mass of the star. To do this, we need the results from exercises 1.3 and 2.1 in *Prialnik*, where the density profile given in a) is also assumed. These are obtained in the following.

From the density we can obtain the total mass, M, as a function of the density at the centre of the star, ρ_c.

$$\begin{aligned} M &= \int_0^r 4\pi r^2 \rho(r) dr \\ &= \int_0^r 4\pi r^2 \rho_c (1 - \frac{r^2}{R^2}) dr \\ &= 4\pi \rho_c \left[\frac{r^3}{3} - \frac{r^5}{5R^2} \right]_0^R \end{aligned}$$

$$M = \frac{8}{15}\pi\rho_c R^3 \tag{7}$$

This can be inverted to give the density at the centre as a function of M:

$$\rho_c = \frac{15M}{8\pi R^3} \tag{8}$$

By integrating the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2}$$

we may determine the pressure at the centre of the star, P_c (assuming, of course, that the star is in hydrostatic equilibrium):

$$P(M) - P(0) = 0 - P_c = -\int_0^R \rho(r)\frac{Gm(r)}{r^2}dr \tag{9}$$

For the density profile given in a), we have (see derivation of M above):

$$m(r) = 4\pi\rho_c\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]_0^R$$

which we can use in equation 9 to obtain P_c:

$$\begin{aligned} P_c &= 4\pi G\rho_c\int_0^R \left(1 - \frac{r^2}{R^2}\right)\left(\frac{r}{3} - \frac{r^3}{5R^2}\right)dr \\ &= 4\pi G\rho_c\int_0^R \frac{r}{3} - \frac{r^3}{3R^2} - \frac{r^3}{5R^2} + \frac{r^5}{5R^4}dr \end{aligned}$$

which, after integration, gives:

$$P_c = \frac{15GM^2}{16\pi R^4} \tag{10}$$

Combining equations 8 and 10 with the state equation for an ideal gas (equation 6), we get the temperature at the centre of the star:

$$\begin{aligned} P_c &= \frac{\mathcal{R}}{\mu}\rho_c T_c \\ T_c &= \frac{\mu}{\mathcal{R}}\frac{P_c}{rho_c} \end{aligned}$$

$$T_c = \frac{1}{2}\frac{\mu G}{\mathcal{R}}\frac{M}{R} \qquad (11)$$

This is almost what we want: the central temperature as a function of M. If ρ_c was a known constant, equation 11 above could be turned into a solution for T_c depending only on M. Inverting equation 7 and substituting in equation 11:

$$T_c = \frac{1}{2}\frac{\mu G}{\mathcal{R}}\frac{M}{\sqrt[3]{\frac{15}{8}\frac{M}{\pi\rho_c}}}$$

$$T_c = \frac{\mu G}{\mathcal{R}}\left(\frac{\pi}{15}\right)^{1/3} M^{2/3}\rho_c^{1/3} \qquad (12)$$

Assuming only the form of the density profile, and that the electron gas in non-degenerate, we can advance further. The assumption of non-degeneracy implies that, for electrons, the ideal gas pressure is higher than the degeneracy pressure, given by (*Prialnik*, section 3.3):

$$\frac{\mathcal{R}}{\mu_e}\rho_c T_c > K_1\left(\frac{\rho_c}{\mu_e}\right)^{5/3}$$

$$\rho_c^2 < \frac{\mathcal{R}^3 T_c^3 \mu_e^2}{K_1^3}$$

$$\rho_c < \sqrt{\frac{\mathcal{R}^3 T_c^3 \mu_e^2}{K_1^3}}$$

with $K_1 = 10^7$ N m^3 kg$^{-5/3}$. Replacing this inequality in 12, we get an upper limit for T_c as a function of the stellar mass M:

$$T_c < \frac{\mu G}{\mathcal{R}}\left(\frac{\pi}{15}\right)^{1/3} M^{2/3}\left(\frac{\mathcal{R}^3 T_c^3 \mu_e^2}{K_1^3}\right)^{1/6}$$

$$T_c < \left(\frac{\pi}{15}\right)^{2/3}\frac{\mu^2\mu_e^{2/3}G^2}{\mathcal{R}}\frac{M^{4/3}}{K_1}$$

Finally, the last equation can be inverted to express a lower limit for the stellar mass necessary for a certain reaction to occur as a function of the the threshold temperature:

$$M^{4/3} > T_c\left(\frac{15}{\pi}\right)^{2/3}\frac{\mathcal{R}K_1}{\mu^2\mu_e^{2/3}G^2}$$

$$M > \sqrt{\frac{15}{\pi}}\frac{\mathcal{R}^{3/4}K_1^{3/4}}{\mu^{3/2}\mu_e^{1/2}G^{3/2}}T_c^{3/4}$$

Nuclear Fuel	Process	$T_{threshold}$ 10^6 K	Ignition mass lower limit 10^{30} kg	Energy per Nucleon (MeV)
H	p-p	~ 4	~ 3.4	6.55
H	CNO	15	9.2	6.25
He	3α	100	38	0.61
C	C+C	600	150	0.54
O	O+O	1000	210	~ 0.3
Si	Nuc.eq.	3000	490	≤ 0.18

Table 2: Major nuclear burning processes and lower limit of masses necessary for central ignition assuming solar composition, non-degeneracy and a density profile given by: $\rho(r) = \rho_c(1 - \frac{r^2}{R^2})$.

substituting the constants in the above expression ($G = 6.672 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $K_1 = 10^7$ N m^3 kg$^{-5/3}$, $\mu_e \sim 1.17$, $\mu \sim 0.61$ and $\mathcal{R} = 8.3145 \times 10^5$ J kg^{-1} K^{-1}), and using the temperature thresholds from table 1 we get:

$$M \quad > \quad 3.81 \times 10^{25} T_c^{3/4} [\mathrm{kg}]$$

This results in the updated table 2.