PH2910 - Discussion Session - Week 6

Ricardo Gonçalo

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1 Polytropic models

The stellar *structure equations*, given by:

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \quad (1)$$

$$\frac{dm}{dr} = 4\pi r^2\rho \quad (2)$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{k\rho}{T^3}\frac{F}{4\pi r^2} \quad (3)$$

$$\frac{dF}{dm} = 4\pi r^2\rho q \quad (4)$$

are too complicated to be solved analytically as a system of differential equations. But if the pressure, P, can be considered independent from temperature, T, then equations 1 and 2 become independent of 3 and 4, and can be solved independently. This is the case with polytropic models.[1]

In polytropic models, the equation of state is given by:

$$P = K\rho^{\gamma} = K\rho^{1+1/n} \quad (5)$$

where K and $\gamma = 1 + \frac{1}{n}$ are constants, and n is the *polytrope index*. Note that, unusually, the pressure given by the equation of state, doesn't depend on temperature. If the star is dominated by the electron degeneracy pressure, its equation of state is given by equation 5. For such a star, the polytropic index is $n = 1.5$ in the case of a non-relativistic degenerate electron gas and

[1] *Polytrope* (meaning something like"many turns") designates a type of function or conic section. Exactly which type depends on the polytrope index n.

to $n = 3$ in the even more extreme case of a relativistic degenerate electron gas. This type of model is relevant in the case of *white dwarves.*

Taking equation 1, it may be multiplied by r^2/ρ and differentiated with respect to r:

$$
\begin{aligned}
\frac{r^2}{\rho}\frac{dP}{dr} &= -Gm \\
\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -G\frac{dm}{dr}
\end{aligned}
$$

Using equation 2:

$$
\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi r^2 G\rho \tag{6}
$$

$$
\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{7}
$$

Now we use equation 5 to express the terms in P as functions of ρ:

$$
\begin{aligned}
\frac{dP}{dr} &= \frac{dP}{d\rho}\frac{d\rho}{dr} \\
&= K\gamma\rho^{\gamma-1}\frac{d\rho}{dr} \\
&= \frac{K(n+1)}{n}\rho^{\frac{1}{n}}\frac{d\rho}{dr} \\
\frac{r^2}{\rho}\frac{dP}{dr} &= \frac{K(n+1)}{n}\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}
\end{aligned}
$$

using the last equation, expression 7 can be written as:

$$
\frac{K(n+1)}{n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -4\pi G\rho
$$

or

$$
\frac{K(n+1)}{4\pi Gn}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho \tag{8}
$$

This is finally a differential equation which describes the dependence of the star's density on the radius. It's a second order equation, so it's solution requires two boundary conditions. The following boundary condictions may be taken:

$$
\begin{aligned}
\rho(r = R) &= 0 \text{ from : } \mathrm{P(r = R) = 0} \\
\left(\frac{d\rho}{dr}\right)_{r=0} &= 0 \text{ from : } \left(\frac{\mathrm{dP}}{\mathrm{dr}}\right)_{\mathrm{r=0}} = 0
\end{aligned}
$$

To solve equation 8 for particular cases of the polytrope index, n, it is useful to perform a substitution using a variable θ defined by:

$$\rho = \rho_c \theta^n \text{ with } 0 \leq \theta \leq 1 \tag{9}$$

with n the same as the polytrope index. If $\theta = 1$, then $\rho = 0$, i.e. a θ of 1 corresponds to the star's surface. At the other end of the interval, $\theta = 0$ and $\rho = \rho_c$, which then corresponds to the centre of the star. Using 9, we have:

$$
\begin{aligned}
\frac{1}{\rho^{\frac{n-1}{n}}} \frac{d\rho}{dr} &= \frac{1}{(\rho_c \theta^n)^{\frac{n-1}{n}}} \frac{d(\rho_c \theta^n)}{dr} \\
&= \frac{1}{\rho_c^{\frac{n-1}{n}} \theta^{n-1}} \rho_c n \theta^{n-1} \frac{d\theta}{dr} \\
&= n \rho_c^{\frac{1}{n}} \frac{d\theta}{dr}
\end{aligned}
$$

Doing the substitution in 8 we have:

$$
\begin{aligned}
\frac{K(n+1)}{4\pi G n} \frac{1}{r^2} \frac{d}{dr} \left(r^2 n \rho_c^{\frac{1}{n}} \frac{d\theta}{dr} \right) &= -\rho_c \theta^n \\
\frac{K(n+1)}{4\pi G} \rho_c^{\frac{1}{n}} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) &= -\rho_c \theta^n \\
\left[\frac{K(n+1)}{4\pi G \rho_c^{\frac{n-1}{n}}} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) &= -\theta^n
\end{aligned}
$$

where the term inside square brackets is a constant with units of length squared which will be written as α^2:

$$\alpha^2 = \left[\frac{K(n+1)}{4\pi G \rho_c^{\frac{n-1}{n}}} \right] \tag{10}$$

and the equation above becomes:

$$\alpha^2 \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) \quad = \quad -\theta^n \tag{11}$$

We'll can now do a further substitution, defined by:

$$r \quad = \quad \alpha\xi \text{ and also } \frac{\mathrm{d}}{\mathrm{dr}} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi} \tag{12}$$

and so

$$\alpha^2 \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) \quad = \quad \frac{\alpha^2}{\alpha^2\xi^2\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$$

which may be replaced in 11 to give:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{13}$$

where $\rho = \rho_c\theta^n$ and $r = \alpha\xi$. Equation 13 is known as the *Lane-Emden* equation. The following boundary conditions may be used in its solution:

$$\begin{aligned} r \quad &= \quad 0 \to \xi = 0 \& \rho = \rho_c \\ \theta(\xi = 0) \quad &= \quad 1 \\ \left(\frac{d\rho}{dr}\right)_{r=0} \quad &= \quad 0 \to \left(\frac{d\rho}{d\xi}\right)_{\xi=0} \end{aligned}$$

Example: $n = 0$ If the polytrope index, n, is zero, then the star density is given by:

$$\rho \quad = \quad \rho_c\theta^0 = \rho_c \equiv constant$$

and the Lane-Emden equation can be written as

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2$$

This equation is now separable and cab be easily integrated in the following steps:

$$\int d\left(\xi^2\frac{d\theta}{d\xi}\right) \quad = \quad -\int \xi^2 d\xi$$

$$\begin{aligned}
\xi^2 \frac{d\theta}{d\xi} &= -\frac{\xi^3}{3} + A \\
\frac{d\theta}{d\xi} &= -\frac{\xi}{3} + \frac{A}{\xi^2} \\
\int d\theta &= -\int \left(\frac{\xi}{3} + \frac{A}{\xi^2} \right) d\xi \\
\theta &= -\frac{\xi^2}{6} - \frac{A}{\xi} + B
\end{aligned}$$

To find the integration constants A and B we can notice that the solution would produce a singularity at $\xi = 0$ for any finite value of A, i.e. A must be zero, otherwise θ becomes infinite at $\xi = 0$. For B, we can notice that at the centre of the star, i.e. for $\xi = 0$, θ must be 1:

$$\begin{aligned}
A &= 0 \\
\theta(\xi = 0) &= 1 \rightarrow B = 1
\end{aligned}$$

So we're left with:

$$\theta = -\frac{\xi^2}{6} + 1$$

This is shown in figure 1. It should be noticed that this is not a realistic model! We have $\rho = \rho_c$, i.e. the density is constant troughout the star. So we get a solution for $\theta(\xi)$ but no dependence of ρ on θ. Non-zero values of n give more realistic models.

It is found that for $n < 5$, $\theta(\xi)$ is monotonic and decreasing with increasing ξ and has a zero at $r = R_{star}$, as was also found for the case $n = 0$ above.

1.1 Central density and average density

The total mass of the star may be written as:

$$\begin{aligned}
M &= \int_0^R 4\pi r^2 \rho dr \\
&= 4\pi \int_0^{\xi_R} (\alpha\xi)^2 (\rho_c \theta^n) \alpha d\xi \\
&= 4\pi\alpha^3 \rho_c \int_0^{\xi_R} \xi^2 \theta^n d\xi
\end{aligned}$$

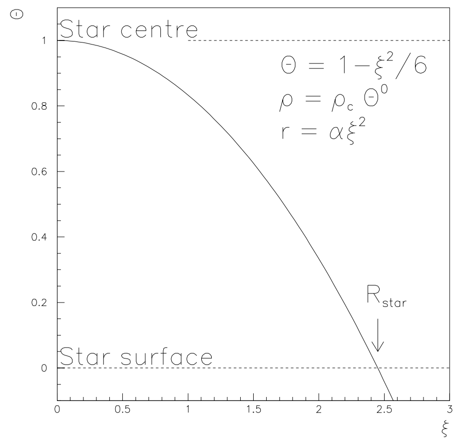

Figure 1: θ versus ξ for an hypothetical star described by a polytrope of index $n = 0$.

where ξ_R is the value of ξ at the star's surface, given by $R = \alpha\xi_R$. But, from equation 13:

$$\xi^2\theta^n = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$$

and the total mass becomes:

$$\begin{aligned} M &= -4\pi\alpha^3\rho_c\int_0^{\xi_R}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \\ &= -4\pi\alpha^3\rho_c\left[\xi^2\frac{d\theta}{d\xi}\right]_0^{\xi_R} \\ &= -4\pi\alpha^3\rho_c\xi_R^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_R} \\ &= -\frac{4\pi R^3}{3}\rho_c\frac{3}{\xi_R}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_R} \end{aligned}$$

As $R = \alpha\xi_R$, this can also be written as:

$$M = \frac{4\pi R^3}{3}\rho_c\left[-\frac{3}{\xi_R}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_R}\right] \quad (14)$$

n	D_n	M_n	R_n
1.0	3.290	3.14	3.14
1.5	5.991	2.71	3.65
2.0	11.40	2.41	4.35
2.5	23.41	2.19	5.36
3.0	52.18	2.02	6.90

Table 1: Constants of polytrope models for different indices n (see *Prialnik* table 5.1).

The first term on the right-hand side is the volume of the star. The term in square brackets is a constant which depends only on the polytrope index. It is usually written as D_n^{-1}:

$$D_n = \left[-\frac{3}{\xi_R} \left(\frac{d\theta}{d\xi} \right)_{\xi=\xi_R} \right]^{-1}$$

This and other constants for polytrope mnodels may be found in table 1 or in *Prialnik* table 5.1. Inverting equation 14, we get an expression for the central density of an star described by a polytrope of index n, which then depends linearly on the average density of the star, $\overline{\rho}$:

$$\begin{aligned} \rho_c &= \frac{M}{\frac{4\pi R^3}{3}} D_n \\ \rho_c &= \overline{\rho} D_n \end{aligned} \tag{15}$$

1.2 Mass-radius relation

Going back to a step in the derivation of equation 14,

$$\begin{aligned} M &= -4\pi\alpha^3 \rho_c \xi_R^2 \left(\frac{d\theta}{d\xi} \right)_{\xi=\xi_R} \\ M &= 4\pi\alpha^3 \rho_c M_n \end{aligned}$$

where M_n is a constant given by:

$$M_n = -\xi_R^2 \left(\frac{d\theta}{d\xi} \right)_{\xi=\xi_R}$$

and is also listed in table 1. This can be further written as:

$$\begin{aligned} \frac{M}{M_n} &= 4\pi\alpha^3\rho_c \\ \left(\frac{GM}{M_n}\right)^{n-1} &= G^{n-1}(4\pi\alpha^3)^{n-1}\rho_c^{n-1} \end{aligned} \tag{16}$$

Now, solving equation 10 for ρ_c, we have:

$$\rho_c = \left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{\frac{n}{n-1}}$$

and we can substitute ρ_c in equation 16 to get:

$$\begin{aligned} \left(\frac{GM}{M_n}\right)^{n-1} &= G^{n-1}(4\pi\alpha^3)^{n-1}\left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^n \\ &= \frac{G^{n-1}}{G^n}\frac{(4\pi)^{n-1}}{(4\pi)^n}\frac{\alpha^{3n-3}}{\alpha^{2n}}K^n(n+1)^n \\ &= \frac{1}{4\pi G}\alpha^{n-3}\left(K(n+1)\right)^n \\ &= \frac{1}{4\pi G}\left(\frac{R}{\xi_R}\right)^{n-3}[K(n+1)]^n \end{aligned}$$

where the relation $\alpha = R/\xi_R$ was used in the last step. ξ_R is a constant which depends only on the polytrope index and is normally written R_n. It can be found in table 1. This leads finally to a relation between the mass and the radius of a star described by a polytrope of index n:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G} \tag{17}$$

Note that the right-hand side is a constant depending only on the polytrope index.

1.2.1 Special cases

Looking at equation 17, we can see that two special cases exist in the mass-radius relation. For $n = 3$ we get:

$$\left(\frac{GM}{M_3}\right)^2\left(\frac{R}{R_3}\right)^0 = \frac{[4K]^3}{4\pi G}$$

$$\left(\frac{GM}{M_3}\right)^2 = \frac{4^2K^3}{\pi G}$$

$$M^2 = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{\frac{3}{2}} \tag{18}$$

i.e., there is only one value of the star's mass, M, which satisfies hydrostatic equilibrium! For $n = 3$, M doesn't depend on R at all!

A second special case exists, clearly, for $n = 1$. In this case, the dependence of R on M disappears and we get:

$$\left(\frac{GM}{M_1}\right)^0 \left(\frac{R}{R_1}\right)^2 = \frac{2K}{4\pi G}$$

$$\left(\frac{R}{R_1}\right)^2 = \frac{2K}{4\pi G}$$

$$R = R_1\sqrt{\frac{2K}{4\pi G}}$$

i.e., for $n = 1$ the star's radius is fixed, independently of the star's mass.

For the intermediate cases of $1 < n < 3$, the radius (raised to a positive power) is proportional to the inverse of the mass (also raised to a positive power):

$$R^{3-n} \propto \frac{1}{M^{n-1}} \tag{19}$$

2 The Chandrasekhar mass

As was said in the beginning of section 1, white dwarves are very dense stars where the pressure is dominated by the electron degeneracy pressure. The relevant equation of state in the case of white dwarves corresponds to a polytropic model with polytrope index of $n = 1.5 = 3/2$ for the non-relativistic case, and of $n = 3$ for the relativistic case, where the thermal electron velocity becomes non-negligible with respect to the velocity of light.

Let's assume initially that the star is non-relativistic, and so the polytrope index is $n = 3/2$. The proportionality constant for this case is $K = K_1' = 10^7 \frac{Nm^{-2}}{(kgm^{-3})^{5/3}}$ (see *Prialnik* sec.3.3). From equation 5, we get a pressure proportional to $\rho^{5/3}$. Also, from equation 19, we get:

$$R^{3-\frac{3}{2}} \propto \frac{1}{M^{\frac{3}{2}-1}}$$
$$M \propto \frac{1}{R^3}$$

If we now imagine similar stars but with increasingly large masses, we see that their radiae will be smaller as the masses are larger.

For this case, we get a density which depends on the square of the mass of the star:

$$\rho \propto \frac{M}{V} \propto \frac{M}{R^3} \propto M^2$$

As we go to higher star masses, the density grows so much that the electron gas occupies all the available states up to high values of energy, and many electrons become relativistic. As the density increases, the equation of state becomes closer to a polytropic model with an index $n = 3$. The pressure is, then, proportional to $\rho^{4/3}$. The proportionality constant is $K = K_2' = 1.24 \times 10^{10} \frac{Nm^{-2}}{(kgm^{-3})^{4/3}}$, given by the expression[2]:

$$K_2' = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_H^{4/3} \mu_e^{4/3}}$$

We have seen in section 1.2.1 that for the case $n = 3$ there is only one possible solution for the mass. This is the limit value that is possible in a compact star dominated by the electron degeneracy pressure. This mass is known as the Chandrasekhar mass, M_{Ch}. In this case, the mass of the star, and is given by equation 18 with K replaced by K_2':

$$M_{Ch} = \frac{M_3\sqrt{3/2}}{4\pi} \left(\frac{hc}{Gm_H^{4/3}}\right)^{3/2} \frac{1}{\mu_e^2}$$

For a hydrogen-depleted star, we have $\mu_e = 2$ (for a star made essentially of iron $\mu_e = 2.15$) and the Chandrasekhar mass becomes:

$$M_{Ch} = 1.46 M_\odot$$

This is the highest possible mass for a star dominated by the electron degenearcy pressure and that is still in hydrostatic equilibrium.

[2]See *Prialnik* section 3.3.