PH2910 - Discussion Session - Week 8

The Virial Theorem and Secular Stability

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1 The Virial Theorem

The virial theorem conects the star's internal energy (and so the pressure and density) with the star's gravitational energy. This connection provides a feedback mechanism which is responsible for the star's *secular* stability.

1.1 The virial theorem

For a star of total mass M and radius R, with pressure and density functions $P(m)$ and $\rho(m)$ of the integrated mass variable m, the virial theorem in a star may be stated as:

$$\Omega = -3\int_0^M \frac{P}{\rho} dm \tag{1}$$

Ω is the gravitational potential energy:

$$\Omega = -\int_0^M \frac{Gm}{r} dm \tag{2}$$

where r is the distance from the centre of the star (which is also a function of m). Note that Ω is negative, as the star is a bound system. The relationship between P and ρ depends on which equation of state describes the gas.

1.2 Derivation of the virial theorem

From the hydrostatic equilibrium equation

$$\frac{dP}{dm} = \frac{-Gm}{4\pi r^4}$$

Multiply both sides by the volume, $V = \frac{4}{3}\pi r^3$ and integrate to get:

$$\begin{aligned} \int_{P(0)}^{P(R)} V dP &= -\int_0^M \frac{Gm}{4\pi r^4}\frac{4}{3}\pi r^3 dm \\ \int_{P(0)}^{P(R)} V dP &= -\frac{1}{3}\int_0^M \frac{Gm}{r} dm \end{aligned}$$

Now, the right-hand side can be identified as the gravitational potential energy (see equation 2) divided by three. The left-hand side can be integrated by parts:

$$\begin{aligned} \int_{P(0)}^{P(R)} V dP &= [VP]_{P(0)}^{P(R)} - \int_{P(0)}^{P(R)} P\frac{dV}{dP} dP \\ &= [VP]_{P(0)}^{P(R)} - \int_{P(0)}^{P(R)} P dV \\ &= V(R)P(R) - V(0)P(0) - \int_{P(0)}^{P(R)} P\frac{dV}{dm} dm \end{aligned}$$

The first two terms vanish because the pressure is zero at the star's surface and the volume is zero at the star's geometric centre. As $dm = \rho dV$, we are left with:

$$-\int_{P(0)}^{P(R)} \frac{P}{\rho} dm = \frac{\Omega}{3}$$

which is the same as equation 1 above.

2 Internal energy

The internal energy of the star is the kinetic energy of the gas particles, including the normal gas, constituted by ions and electrons, and the photon gas. Here, we are considering the cases where one of these contributions is dominant and the others can be safely neglected.

The internal energy can be found by integrating the kinetic energy, $\epsilon(p)$, of the gas particles for the entire range of possible momenta p, weighted by the number of particles having that value of momentum, $n(p)$:

$$\int_0^\infty n(p)\epsilon(p)dp$$

The *specific* energy density of the gas, u_{gas} (i.e., the energy per unit mass and unit volume) can be obtained from the expression above by dividing by the density:

$$u_{gas} = \frac{1}{\rho}\int_0^\infty n(p)\epsilon(p)dp \tag{3}$$

In the following, we'll also need to know the *pressure integral*, which gives the pressure for a certain number distribution $n(p)$ of particles with momentum p. This can be written[1]:

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp \tag{4}$$

2.1 Classical ideal gas

For an classical ideal gas, the energy is given by the familiar expression $\epsilon = \frac{p^2}{2m}$, and the momentum distribution, $n(p)$, is the *Maxwell-Boltzmann distribution*:

$$n(p) = \frac{n_I 4\pi p^2}{(2\pi m_I kT)^{3/2}} e^{\frac{-p^2}{2m_I kT}} \tag{5}$$

where n_I is the total number of ions per unit volume, m_I is the mean molecular mass, $k = 1.380658 \times 10^{-23}\mathrm{JK}^{-1}$ is *Boltzmann's constant*, and T is the temperature.

Solving the integral in equation 3 with $n(p)$ given by equation 5 gives[2]:

$$u_{gas} = \frac{3}{2}\frac{1}{\rho}n_I kT \tag{6}$$

[1] See *Prialnik* section 3.1

[2] See section A.1.

On the other hand, the equation of state for a classical ideal gas is: $P_{gas} = nkT$ (obtained using the pressure integral, equation 4 above[3]). This gives:

$$u_{gas} = \frac{3}{2}\frac{P_{gas}}{\rho}$$

Finally, integrating over the whole star, we get the total internal energy of the star's gas, U_{gas}:

$$\begin{aligned} U_{gas} &= \int_0^M u_{gas} dm \frac{3}{2}\frac{P_{gas}}{\rho} \\ &= \frac{3}{2}\int_0^M \frac{P_{gas}}{\rho} dm \end{aligned} \tag{7}$$

Now, using the virial theorem, we have:

$$\Omega = -3\int_0^M \frac{P}{\rho} dm$$

But, from equation 7, the right-hand side is simply twice the internal energy:

$$\Omega = -2U_{gas} \tag{8}$$

2.2 Non-negligible radiation pressure

If the contribution of the (photon gas) radiation to the total pressure cannot be neglected, we must include it in the expression for $\frac{P}{\rho}$:

$$\frac{P}{\rho} = \frac{P_{gas} + P_{rad}}{\rho}$$

How do we calculate P_{rad}? The same way as above, except the momentum distribution is now given by *Planck's black-body distribution* instead of the Maxwell-Boltzmann distribution, and the energy ϵ is now given by $\epsilon = h\nu$, where h is Planck's constant. The black-body distribution is given in terms of the frequency, ν, by:

$$n(\nu) = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{9}$$

[3] The detailed calculation of the integral is shown in section A.2

where c is the speed of light. Using the above expression in equation 4, and taking into account $v = c$ and $p = \frac{h\nu}{c}$, we get:

$$P_{rad} = \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}\frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}d\nu$$

The solution of this integral is shown in section A.3. The result is (equation A.3):

$$P_{rad} = \frac{1}{3}aT^4$$

where the a is the *radiation constant*: $a = \frac{8\pi^5 k^4}{15c^3h^3}$.

In a similar way, using equation 3 and the black-body distribution, equation 9, we get for the specific internal energy of the photon gas (see section A.4 and equation 16):

$$u_{rad} = \frac{aT^4}{\rho} = 3\frac{P_{rad}}{\rho}$$

which results in the total internal energy arising from the photon gas:

$$U_{rad} = 3\int_0^M \frac{P_{rad}}{\rho}dm = \frac{P_{rad}}{\rho} \tag{10}$$

Now, for non-negligible radiation pressure, the virial theorem (eq. 1) can be written:

$$\begin{aligned}
\Omega &= -3\int_0^M \frac{P}{\rho}dm \\
\Omega &= -3\int_0^M \frac{P}{\rho}dm - 3\int_0^M \frac{P}{\rho}dm \\
\Omega &= -2U_{gas} - U_{rad} \qquad (11)
\end{aligned}$$

where equations 8 and 10 were used in the last step.

3 Secular Stability

The total energy of the star, E, is the sum of the gravitational potential energy and of the gas and radiation internal energy. If radiation pressure is negligible this means:

$$\begin{aligned} E &= \Omega + U_{gas} \\ &= -2U_{gas} + U_{gas} \end{aligned}$$

(see equation 8). Including the radiation pressure, we have:

$$\begin{aligned} E &= \Omega + U_{rad} + U_{gas} \\ E &= -2U_{gas} - U_{rad} + U_{rad} + U_{gas} \end{aligned}$$

(see equation 11). Both the above equations result in:

$$E = -U_{gas} \tag{12}$$

i.e. the total energy is in both cases the negative of the total internal energy of the gas.

We can now consider the physics of the system. If the star becomes smaller, the gravitational potential energy, Ω, increases in value (i.e., becomes more negative), as it depends on the inverse of the radius (see equation 2). From equations 11 and 8, the (positive) internal energy of the gas, U_{gas}, increases. This means that the star temperature increases. A higher temperature means a higher average gas particle momentum and so a higher internal gas energy. From equation 12, the total energy, E, also increases in value and becomes more negative.

It the reverse situation, if the star expands, Ω decreases, which leads to smaller U_{gas} and so smaller star temperature and a total energy closer to zero, i.e. "less negative".

The previous paragraphs explain the ingredients of the star's stability. Let's consider what happens if there's a sudden increase of the rate of nuclear burning. For a star, we have:

$$\dot{E} = L_{nuclear} - L$$

where L is the star's luminosity and $L_{nuclear}$ is the energy produced by nuclear fuel burning. In thermal equilibrium, $L_{nuclear}$ is equal to L. If, on the other hand, there's a perturbation which causes an increase of the energy generated in nuclear reactions, then $L_{nuclear} > L$ and therefore $\dot{E} > 0$. Since E is negative (as the star is a bound system), then $|E|$ becomes smaller. But from equation 12, this causes U_{gas} to *decrease* and so also the temperature to decrease.

This means that a perturbation which increases the nuclear burning rate actually causes a decrease in the star's temperature. This can be described as the star having a negative heat capacity. I.e., we supply heat to the star in the form of increased nuclear burning and this has the effect of decreasing the star's temperature.

As the rate of nuclear reactions varies strongly with the temperature, a small decrease in the temperature leads to a much smaller rate of nuclear burning, and so the initial perturbation disapears.

The same type or argument applies to an initial perturbation which decreases the rate of nuclear burning. In this case, the star becomes hotter, and the nuclear burning rate is increased, which opposes the trend of the initial perturbation.

A Appendix: Some detailed calculations

A.1 Ideal gas internal energy

Solving the integral 3 with $n(p)$ given by 5 and the energy given by $\epsilon = \frac{p^2}{2m}$ gives:

$$\begin{aligned} u_{gas} &= \frac{1}{\rho}\int_0^\infty \frac{n_I 4\pi p^2}{(2\pi m_I kT)^{3/2}} e^{\frac{-p^2}{2m_I kT}} \frac{p^2}{2m_I} dp \\ &= \frac{1}{\rho}\frac{n_I 4\pi}{(2\pi m_I kT)^{3/2}} \int_0^\infty e^{\frac{-p^2}{2m_I kT}} \frac{p^4}{2m_I} dp \\ &= \frac{1}{\rho}\frac{n_I 4\pi}{(2\pi m_I kT)^{3/2}} \frac{m_I kT}{2m_I} \int_0^\infty e^{\frac{-p^2}{2m_I kT}} p^3 d\left(\frac{p^2}{2m_I kT}\right) \end{aligned} \tag{13}$$

In the last step, the flowing substitution was made:

$$d\frac{p^2}{2m_I kT} = \frac{d}{dp}\frac{p^2}{2m_I kT}dp = \frac{2p}{2m_I kT}dp$$

We can try to transform it into an integral of the form $\int_0^\infty e^{-x}x^n dx$:

$$\begin{aligned} u_{gas} &= \frac{1}{\rho}\frac{n_I 2\pi kT}{(2\pi m_I kT)^{3/2}} \int_0^\infty e^{\frac{-p^2}{2m_I kT}} \left(\frac{p^2}{2m_I kT}\right)^{3/2} (2m_I kT)^{3/2} d\left(\frac{p^2}{2m_I kT}\right) \\ &= \frac{1}{\rho}\frac{n_I 2\pi kT (2m_I kT)^{3/2}}{(2\pi m_I kT)^{3/2}} \int_0^\infty e^{\frac{-p^2}{2m_I kT}} \left(\frac{p^2}{2m_I kT}\right)^{3/2} d\left(\frac{p^2}{2m_I kT}\right) \end{aligned}$$

$$= \frac{1}{\rho}\frac{n_I 2\pi kT}{\pi^{3/2}}\int_0^\infty e^{-y}y^{3/2}dy$$

with $y = \frac{p^2}{2m_I kT}$. The integral may be solved by integrating twice by parts. Note that the constant term is zero in both integrations by parts:

$$\begin{aligned}
\int_0^\infty e^{-y}y^{3/2}dy &= -\left[e^{-y}y^{3/2}\right]_0^\infty + \frac{3}{2}\int_0^\infty e^{-y}y^{1/2}dy \\
&= 0 + \frac{3}{2}\int_0^\infty e^{-y}y^{1/2}dy \\
&= \frac{-3}{2}\left[e^{-y}y^{1/2} + \frac{3}{2}\right]_0^\infty \int_0^\infty e^{-y}y^{-1/2}dy \\
&= 0 + \frac{3}{2}\int_0^\infty e^{-y}y^{-1/2}dy
\end{aligned}$$

Making an additional substitution $y = z^2$, we have $dy = \frac{dy}{dz}dz = 2zdz$ and:

$$\begin{aligned}
\int_0^\infty e^{-y}y^{-1/2}dy &= \int_0^\infty e^{-z^2}z^{-1}2zdz \\
&= \int_0^\infty e^{-z^2}dz
\end{aligned}$$

This doesn't look like much of an improvement, but in fact this integral is simply $\sqrt{\pi}/2$, according to equation 20 in section A.5.3.

This leaves us with:

$$\begin{aligned}
u_{gas} &= \frac{1}{\rho}\frac{n_I 2\pi kT}{\pi^{3/2}}\frac{3}{2}\frac{\sqrt{\pi}}{2} \\
&= \frac{1}{\rho}\frac{3}{2}\frac{n_I \pi^{3/2} kT}{\pi^{3/2}} \\
&= \frac{1}{\rho}\frac{3}{2}n_I kT
\end{aligned}$$

A.2 Ideal gas pressure

The ideal gas pressure can be obtained from the pressure integral (equation 4) and the Maxwell-Boltzmann distribution (equation 5):

$$P_{gas} = \frac{1}{3}\int_0^\infty vpn(p)dp$$

$$
\begin{aligned}
&= \frac{1}{3}\int_0^\infty \frac{n_I 4\pi p^2}{(2\pi m_I kT)^{3/2}} e^{\frac{-p^2}{2m_I kT}} vp dp \\
&= \frac{1}{3}\int_0^\infty \frac{n_I 4\pi p^2}{(2\pi m_I kT)^{3/2}} e^{\frac{-p^2}{2m_I kT}} \frac{p^2}{m} dp
\end{aligned}
$$

But the last equation is almost exactly the same as equation 13. The difference is a multiplying factor of $2\rho/3$, i.e.:

$$
P_{gas} = \frac{2\rho}{3}\frac{1}{\rho}\int_0^\infty \frac{n_I 4\pi p^2}{(2\pi m_I kT)^{3/2}} e^{\frac{-p^2}{2m_I kT}} \frac{p^2}{2m_I} dp
$$

So the solution of the iontegral is simple:

$$
\begin{aligned}
P_{gas} &= \frac{2\rho}{3} u_{gas} \\
P_{gas} &= \frac{2\rho}{3}\frac{1}{\rho}\frac{3}{2} n_I kT \\
P_{gas} &= n_I kT
\end{aligned}
$$

A.3 Radiation pressure

The radiation pressure, produced by the photon gas, can be calculated using the pressure integral, equation 4, and Planck's black-body distribution, equation 9 (as this is given in terms of the frequency, ν, this is the integrating variable that we'll use). The velocity is c and the momentum is $p = h/\lambda = h\nu/c$:

$$
\begin{aligned}
P_{rad} &= \frac{1}{3}\int_0^\infty c\frac{h\nu}{c}\frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu \\
&= \frac{1}{3}\frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} d\nu \qquad (14) \\
&= \frac{1}{3}\frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{y^3}{e^y - 1} dy
\end{aligned}
$$

In the last step, the following substitution was made: $y = \frac{h\nu}{kT}$, $dy = \frac{h}{kT} d\nu$.

The integral above is evaluated in section A.5.4 (see equation 21) and is simply equal to $\pi^4/15$. The radiation pressure is then:

$$
P_{rad} = \frac{1}{3}\frac{8\pi h^2}{c^4}\left(\frac{kT}{h}\right)^4 \frac{\pi^4}{15}
$$

$$\begin{aligned} &= \frac{1}{3}\frac{8\pi^5 k^4}{15c^3h^3}T^4 \\ &= \frac{1}{3}aT^4 \end{aligned} \tag{15}$$

where $a = \frac{8\pi^5 k^4}{15c^3h^3}$ is called the radiation constant.

A.4 Photon gas internal energy

The internal energy of the photon gas is given by equation 3 with n given by Planck's black-body distribution, equation 9 and the energy given by $\epsilon = h\nu$. Using the frequency as integrating variable:

$$\begin{aligned} u_{gas} &= \frac{1}{\rho}\int_0^\infty \frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}h\nu d\nu \\ &= \frac{1}{\rho}\frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}}-1}d\nu \end{aligned}$$

The last equation is very similar to equation 14. Comparing the two, we immediately obtain:

$$u_{gas} = \frac{1}{\rho}aT^4 \tag{16}$$

with $a = \frac{8\pi^5 k^4}{15c^3h^3}$.

A.5 Evaluation of some useful integrals

A.5.1 Evaluation of $\int_0^\infty e^{-x}x^n dx$

For $n = 0$ the solution is trivial:

$$\begin{aligned} \int_0^\infty e^{-x}dx &= -\left[e^{-x}\right]_0^\infty \\ &= -[0-1] = 1 \end{aligned} \tag{17}$$

For $n > 0$ we can evaluate the integral by parts:

$$\begin{aligned} \int_0^\infty e^{-x}x^n dx &= -\left[e^{-x}x^n\right]_0^\infty + n\int_0^\infty e^{-x}x^{n-1}dx \\ &= -0 + 0 + n\int_0^\infty e^{-x}x^{n-1}dx \\ \int_0^\infty e^{-x}x^n dx &= n\int_0^\infty e^{-x}x^{n-1}dx \end{aligned} \tag{18}$$

Equation 18 is a recurrence relation that can be used to calclate the integral. Since we've got the integral for $n = 0$, from equation 17, we can in principle evaluate all others. Incidently, if n is a positive integer:

$$\int_0^\infty e^{-x} x^n dx \quad = \quad n(n-1)(n-2)...1 = n! \tag{19}$$

A.5.2 Evaluation of $\int_0^\infty e^{-\alpha x} x^n dx$

This is easily obtained from the results of the previous section:

$$\begin{aligned} \int_0^\infty e^{-\alpha x} x^n dx &= \frac{1}{\alpha^n} \int_0^\infty e^{-\alpha x} (\alpha x)^n dx \\ &= \frac{1}{\alpha^{n+1}} \int_0^\infty e^{-\alpha x} (\alpha x)^n d\alpha x \end{aligned}$$

and we do an obvious substitution, $y = \alpha x$, to get:

$$\int_0^\infty e^{-\alpha x} x^n dx \quad = \quad \frac{1}{\alpha^{n+1}} \int_0^\infty e^{-y} y^n dy$$

A.5.3 Evaluation of $\int_0^\infty e^{-x^2} dx$

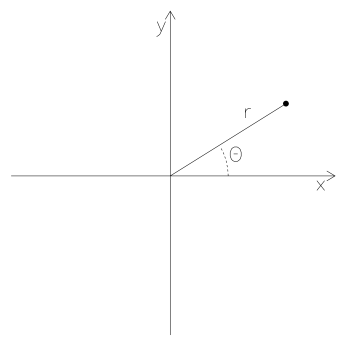
This integral can be solved using a little trick. Let's write:

$$I \quad = \quad \int_{-\infty}^\infty e^{-x^2} dx$$

and let's evaluate I^2. Note that the integration variables are independent in the two integrals, we'll use x and y to distinguish them.

$$\begin{aligned} I^2 &= \int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2} e^{-y^2} dy dx \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dy dx \end{aligned}$$

Now, x and y may be considered as independent geometrical co-ordinates in a plane as shown in the figure. This is equivalent to making the usual variable substitution when converting from cartesian to polar co-ordinates.



The integral can, then, be written as:

$$
\begin{aligned}
I^2 &= \int_0^\infty \int_0^{2\pi} e^{-r^2} r d\theta dr \\
&= 2\pi \int_0^\infty e^{-r^2} dr \\
&= 2\pi \int_0^\infty -\frac{1}{2} d\left(e^{-r^2}\right) \\
&= -\pi \int_0^\infty d\left(e^{-r^2}\right) \\
&= -\pi \left[e^{-r^2}\right]_0^\infty = -\pi\,[0-1] = \pi
\end{aligned}
$$

i.e.

$$I = \sqrt{\pi}$$

and, since e^{-x^2} is an even function:

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}/2 \qquad (20)$$

A.5.4 Evaluation of $\int_0^\infty x^3 \frac{1}{e^{-x}-1} dx$

This integral can be evaluated by expanding the integrand in a series. Replacing e^{-x} with z, we can write[4]:

$$\frac{x^3}{e^{-x}-1} = \frac{x^3 e^{-x}}{1-e^{-x}} = x^3 e^{-x} \frac{1}{1-e^{-x}} = x^3 e^{-x} \frac{1}{1-y}$$

Using Taylor's formula, we can now expand $\frac{1}{1-y}$. Taylor's series about zero can be written:

$$f(z) = f(0) + z f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{z^n}{n!} f^{(n)}(0) + ...$$

so we must calculate each term, and evaluate the derivatives at $z = 0$:

$$\begin{aligned}
f(0) &= \left[\frac{1}{1-z}\right]_{z=0} = 1 \\
f'(0) &= \left[\frac{1}{(1-z)^2}\right]_{z=0} = 1 \\
f''(0) &= \left[\frac{2 \times 1}{(1-z)^3}\right]_{z=0} = 2! \\
f'''(0) &= \left[\frac{3 \times 2}{(1-z)^4}\right]_{z=0} = 3! \\
f''''(0) &= \left[\frac{4 \times 3 \times 2}{(1-z)^5}\right]_{z=0} = 4! \\
f^{(n)}(0) &= \left[\frac{n!}{(1-z)^{n+1}}\right]_{z=0} = n!
\end{aligned}$$

This gives the integrand in a series form:

$$\begin{aligned}
\frac{x^3}{e^{-x}-1} &= \frac{x^3 e^{-x}}{1-e^{-x}} = \frac{x^3 e^{-x}}{1-z} \\
&= x^3 e^{-x} (1 + z + \frac{2!}{2!} z^2 + \frac{3!}{3!} z^3 + ...) \\
&= x^3 (e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + ...) \\
&= \sum_{n=1}^{\infty} e^{-nx} x^3
\end{aligned}$$

[4] The interval of convergence for this series is actually $|z| < 1$, which is satisfied by e^{-x}

The integral then becomes:

$$\begin{aligned}
\int_0^\infty \frac{x^3}{e^{-x}-1}dx &= \sum_{n=1}^{\infty}\int_0^\infty e^{-nx}x^3dx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^\infty e^{-nx}(nx)^3dnx \\
&= \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^\infty e^{-}yy^3dy
\end{aligned}$$

Where the following substitution was made: $y = nx$, $dy = ndx$. But, according to equation 19, the integral is simply $3! = 6$, which gives:

$$\int_0^\infty \frac{x^3}{e^{-x}-1}dx = 6\sum_{n=1}^{\infty}\frac{1}{n^4}$$

This is a converging series which can be found to be $\pi^4/90$, and we finally have:

$$\int_0^\infty \frac{x^3}{e^{-x}-1}dx = \frac{\pi^4}{15}$$