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PH4442 Advanced Particle Physics Week 1

Organisational matters → slides

Prelude: recap ~ all of PP

Cast of characters:

spin-½ fermions
+ their antiparticles

units of proton charge $e > 0$

Q				
0	ν_e	ν_μ	ν_τ	leptons
−1	e^-	μ^-	τ^-	
+2/3	u	c	t	quarks
−1/3	d	s	b	

spin-1 "gauge bosons": $\underbrace{\gamma}_{\text{EM}}, \underbrace{Z, W^\pm}_{\text{weak}}, \underbrace{g}_{\text{strong}}$ ← gluon

spin-0 Higgs boson H

+ Quantum Mechanics ~ ptcls created/destroyed
→ QFT

+ Relativity ~ special (no gravity)

+ Gauge symmetry ~ determines allowed interactions

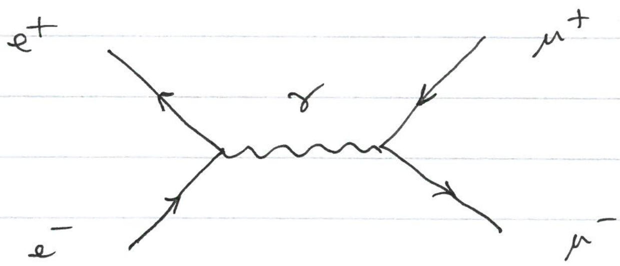
+ 25 free parameters ~ ptcl masses, couplings

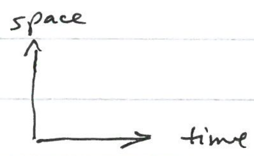
→ <u>Standard Model</u> of PP
↑ "SM"

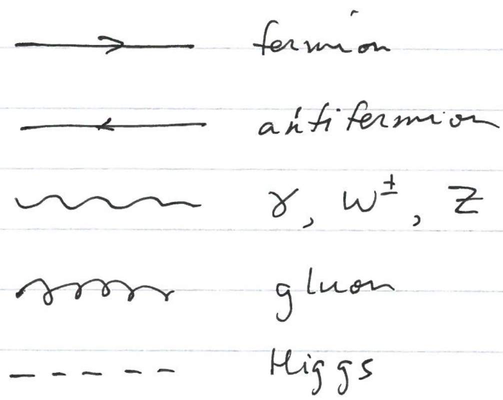
Theoretical Framework

Theory defined by Lagrangian / Eq. of motion; to solve, need Perturbation theory → <u>Feynman diagrams</u> = space-time diagrams that show how particles interact.

e.g.







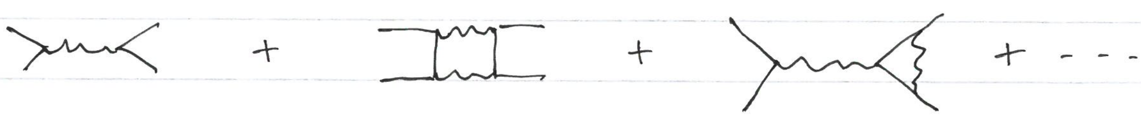
"Feynman rules" give QM amplitude $\mathcal{M}$ for each diagram

If initial → final via multiple intermediate paths, total amplitude = sum

$$\mathcal{M} = \sum_i \mathcal{M}_i$$

← all intermediate paths

e.g.



Probability (rate, cross section) $\sim |\mathcal{M}|^2$

↑ total amp.

PP reactions have speeds $\sim c$

$\Rightarrow$ need relativity

Two big changes relative to non-rel. QM:

1) Schrödinger eq. $\rightarrow$ $\begin{cases} \text{Klein-Gordon eq. (bosons)} \\ \text{Dirac eq. (fermions)} \end{cases}$

2) Particles can be created / destroyed

QM $\rightarrow$ Quantum Field Theory (QFT)

In this course, start w/ relativistic wave eqs., then take some shortcuts to ptcl creation / destruction w/ <u>Feynman - Stückelberg</u> approach (propagator theory) $\rightarrow$ Feynman rules.

Later in course bring in QFT (needed for treatment of ptcl <u>masses</u> in theories w/ local gauge symmetry (Higgs mechanism)

<u>Units</u>: $\hbar = c = 1$

e.g. m means mc^2 $[m] = \text{GeV}$

To convert, insert factors $c = 3.00 \times 10^8 \frac{m}{s}$, $\hbar c = 0.197 \text{ GeV} \cdot \text{fm}$

Relativity

Write point in time t + space $\vec{r} = (x, y, z)$ ← or $\vec{x}$

as **four-vector** $x = (t, x, y, z) = (x^0, x^1, x^2, x^3)$

i.e. components are x^μ, $\underbrace{\mu = 0, \ldots, 3}_{\text{Lorentz index}}$

x^μ = (component of) **contravariant** four-vector
↑ upper-index

x_μ = (" ") **covariant** " " ~ lower index

$= (t, -x, -y, -z) = (x^0, -x^1, -x^2, -x^3)$

$= g_{\mu\nu} x^\nu$ ~ sum over repeated indices

i.e. $\sum_\nu g_{\mu\nu} x^\nu$

where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ ~ components $= g_{\mu\nu}$ $= g^{\mu\nu}$ (same)

(Minkowski) metric tensor (sometimes η)

Usually say $g_{\mu\nu}$ = components of g

$g^{\mu\nu}$ = " " g^{-1}

but they are in fact equal

Can always convert between contravariant + covariant by contracting w/ appropriate g

e.g. $\quad x^\mu = g^{\mu\nu} x_\nu$

<u>Lorentz transformation</u> of four-vector x^μ

$\equiv$ ① linear,

② homogeneous, (leaves origin unchanged)

③ leaves $x_\mu x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ unchanged

i.e. $\quad x' = \Lambda x$ ← column vector $\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

4x4 matrix (<u>indep.</u> of x)

or $\quad x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$

row, column. → standard index pos. for Λ

Transpose of Λ: swap row ↔ col but don't change up, down

$$(\Lambda^T)_\nu{}^\mu = \Lambda^\mu{}_\nu$$

standard index pos. for Λ^T

<u>Not</u> considering translations (yet)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \text{(Poincaré)}$$

Lorentz includes boosts, rotations, parity, time reversal.

Lorentz trans $\overset{x' = \Lambda x}{}$ must have $x'_\mu x'^\mu = x_\mu x^\mu$

$$x'^2 = x'_\mu x'^\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} (\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma)$$

$$= \underbrace{g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma}\, x^\rho x^\sigma$$

Should equal $x^2 = g_{\rho\sigma} x^\rho x^\sigma$

$$\Rightarrow \quad \underline{g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}}$$

General requirement for Lorentz trans.

To write in matrix form, use $\Lambda^\mu{}_\rho = (\Lambda^T)_\rho{}^\mu$

$$(\Lambda^T)_\rho{}^\mu g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

follows usual conventions of matrix mult

$$\rightarrow \quad \underline{\Lambda^T g \Lambda = g}$$

Mult. on left by g^{-1}

$$\underbrace{(g^{-1} \Lambda^T g)}_{\Lambda^{-1} = g^{-1}\Lambda^T g} \Lambda = g^{-1} g = I_4$$

4x4 identity matrix

i.e. $(\Lambda^{-1})^\mu{}_\nu = g^{\mu\rho} (\Lambda^T)_\rho{}^\sigma g_{\sigma\nu} = (\Lambda^T)^\mu{}_\nu \equiv \Lambda_\nu{}^\mu$

seems to say $\Lambda^{-1} = \Lambda^T$ ← not true for boosts,

$(\Lambda^T)^\mu{}_\nu$ is not standard index pos. for Λ^T

$\Rightarrow$ $\Lambda_\nu{}^\mu$ is synonym for $(\Lambda^{-1})^\mu{}_\nu$

[skip in lecture?]

The Lorentz Group ← see App. A

Two successive Lorentz trans = single one

$$x' = \Lambda_2 (\Lambda_1 x) = \underbrace{(\Lambda_2 \Lambda_1)}_{\text{still Lorentz?}} x$$

$$(\Lambda_2 \Lambda_1)^T g (\Lambda_2 \Lambda_1) = \Lambda_1^T \underbrace{\Lambda_2^T g \Lambda_2}_{g} \Lambda_1 = \Lambda_1^T g \Lambda_1 = g \quad \checkmark$$

Summary: set of all trans. $x' = \Lambda x$

w/ $\Lambda^T g \Lambda = g$ ≡ Lorentz <u>group</u>

Different Λ → boosts, rotations, parity, time rev.

[If include translations: $x' = \Lambda x + a$

→ inhomogeneous Lorentz group (Poincaré)]

Two disjoint subsets of transformations:

$$\det(\Lambda^T g \Lambda) = \det g$$

$$= \underbrace{\det \Lambda^T}_{\det \Lambda} \cdot \det g \cdot \det \Lambda$$

$\Rightarrow |\det \Lambda|^2 = 1 \Rightarrow \det \Lambda = \pm 1$

$\det \Lambda = +1$: boost, rotation (<u>proper</u> Lor. tr.)

$\det \Lambda = -1$: parity, time rev. (improper " ")

If $\Lambda = \prod_{i=1}^{n} \Lambda_i$ w/ $\det \Lambda_i = +1 \ \forall i$ (all proper)

$\rightarrow \det \Lambda = \prod_{i=1}^{n} \det \Lambda_i = 1$ → <u>stays</u> proper. → arb. comb. of boost, rot has $\det \Lambda = +1$.

Examples of Lorentz trans (w/o proof)

Boost along z: $\quad x' = \Lambda_B x$

$$\Lambda_B = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \beta = v \quad (c=1) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

or equivalently, let $\cosh\omega = \gamma$ $\quad (\sinh\omega = \beta\gamma)$

rapidity

$$\Lambda_B = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}$$

Rotation of angle θ about z axis

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check: $\det \Lambda_B = \det \Lambda_R = +1$ (proper)

Parity (space inversion) $\qquad$ Time reversal

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\det \Lambda_P = \det \Lambda_T = -1$ (improper)

Four - vectors

$a = (a^0, a^1, a^2, a^3)$ or in matrix eq. a column vector

is a four-vector if it transforms same

as $x = (x^0, x^1, x^2, x^3)$ i.e. in transformed

frame w/ $x' = \Lambda x \rightarrow a' = \Lambda a$ i.e.

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

Default four-vector is contravariant (upper index)

$a_\mu = g_{\mu\nu} a^\nu$ defines covariant version

$$a'_\mu = g_{\mu\rho} a'^\rho = g_{\mu\rho} \Lambda^\rho{}_\sigma \underbrace{g^{\sigma\nu} a_\nu}_{a^\sigma} = \Lambda_\mu{}^\nu a_\nu$$

Given 2 four-vectors a, b, "four-vector product"

$$a \cdot b = a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$$

Since $x^2 = x \cdot x = x_\mu x^\mu$ invariant,

$$a' \cdot b' = a'_\mu b'^\mu = g_{\mu\nu} a'^\nu b'^\mu$$

$$= g_{\mu\nu} \Lambda^\nu{}_\rho a^\rho \Lambda^\mu{}_\sigma b^\sigma = g_{\mu\nu} \Lambda^\nu{}_\rho \Lambda^\mu{}_\sigma a^\rho b^\sigma$$

should $= a \cdot b = g_{\rho\sigma} a^\rho b^\sigma$

Yes, if $g_{\mu\nu} \Lambda^\nu{}_\rho \Lambda^\mu{}_\sigma = g_{\rho\sigma}$ ✓ : see e.g. Feynman vol I ch 17

Example $p = (E, \vec{p})$ = "p^μ" $p' = \Lambda p$

$p \cdot p = p_\mu p^\mu = E^2 - \vec{p} \cdot \vec{p} = m^2$ or invariant

Fields and four-vector derivatives ~ will need for relativistic wave eqs.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \quad \text{(covariant)}$$

⚠ lower index ↑ deriv. wrt x^μ

$$\Rightarrow \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) \quad \text{(contravariant)}$$

Consider Lorentz scalar field $\varphi(x)$ i.e. in transformed frame

$$\varphi'(x') = \varphi(x) \quad \sim \text{by def.}$$

Consider $\partial^\mu \varphi$ In transformed frame

$$\partial'^\mu \varphi'(x') = \frac{\partial \varphi'(x')}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial \varphi'(x')}{\partial x_\nu} = \frac{\partial x_\nu}{\partial x'_\mu}\partial^\nu \varphi(x)$$

Use $x'^\mu = \Lambda^\mu{}_\nu x^\nu \rightarrow x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu$

↖ lower indices by contracting w/ g

$$\rightarrow x_\nu = g_{\nu\rho}(\Lambda^{-1})^\rho{}_\sigma g^{\sigma\mu} x'_\mu \;\Big\} \text{ use for } \frac{\partial x_\nu}{\partial x'_\mu} \text{ w/ } (\Lambda^{-1})^\rho{}_\sigma = g^{\rho\alpha}(\Lambda^T)_\alpha{}^\beta g_{\beta\sigma}$$

$$\Rightarrow \partial'^\mu \varphi'(x') = (\Lambda^T)_\nu{}^\mu \partial^\nu \varphi(x) = \Lambda^\mu{}_\nu \varphi(x)$$

i.e. $\partial^\mu \varphi(x)$ transforms as a contravariant 4-vec

Similarly, $\partial_\mu \varphi(x)$ " " " covariant "

$\partial_\mu \partial^\mu \varphi(x)$ " " " Lorentz scalar

$\hookrightarrow \Box$ or $\Box^2 = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)$ = d'Alembertian op.

Exponential form of Lorentz trans. See Sec. 2.6

Relativistic wave equations

I. Maxwell's eqs.

Heaviside - Lorentz units

$\varepsilon_0 = \mu_0 = c = \hbar = 1$

$$\left.\begin{array}{ll} ① & \nabla \cdot \vec{B} = 0 \\ ② & \nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{array}\right\} \text{homogeneous}$$

$$\left.\begin{array}{ll} ③ & \nabla \cdot \vec{E} = \rho \\ ④ & \nabla \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{j} \end{array}\right\} \text{non-} \quad "$$

From $\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B} = \nabla \times \vec{A} \rightarrow$ use in ②

(swap $\nabla \times$ + $\frac{\partial}{\partial t}$)

$$\Rightarrow \quad \nabla \times \underbrace{\left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right)} = 0$$

$\hookrightarrow$ zero curl $\Rightarrow$ (neg.) gradient of scalar pot. ϕ

$$\Rightarrow \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

$$\left.\begin{array}{ll} \text{Four-vector pot} & A^\mu = (\phi, \vec{A}) \\ \quad " \quad \text{current} & j^\mu = (\rho, \vec{j}) \end{array}\right\} \text{Not completely obvious that these are four-vectors (but true).}$$

Potentials not unique - can change by

<u>gauge trans</u> $\quad A'^\mu = A^\mu - \partial^\mu \chi$ on arb. (Lorentz) scalar func. of $x = (t, \vec{x})$

$\rightarrow$ get same $\vec{E}, \vec{B}$

$\hookrightarrow$ extend this symmetry to weak, strong int.

$\rightarrow$ key principle in constructing SM.

To write Maxwell's eqs. in 4-vec form, define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$ ← field strength tensor

$$= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$ ← antisym: $F^{\mu\nu} = -F^{\nu\mu}$

① + ② : $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$

③ + ④ : $\partial_\mu F^{\mu\nu} = j^\nu$ ← apply ∂_ν

$$\to \partial_\nu j^\nu = \underbrace{\partial_\nu \partial_\mu}_{\substack{\text{sym. under} \\ \mu \leftrightarrow \nu}} \underbrace{F^{\mu\nu}}_{\text{antisym.}} = 0$$

$$\Rightarrow \partial_\nu j^\nu = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$ ← continuity eq.

Put def. of $F^{\mu\nu}$ into $\partial_\mu F^{\mu\nu} = j^\nu$

$$\to (\partial_\mu \partial^\mu) A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu$$

+ consider gauge trans $A'^\mu = A^\mu - \partial^\mu \chi$

$$\Rightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu - \partial_\mu \partial^\mu \chi$$

Can choose χ such that $\partial_\mu A^\mu = \partial_\mu \partial^\mu \chi$

$$\Rightarrow \partial_\mu A^\mu = 0 \quad \text{(drop prime)}$$

↑ Lorenz gauge

In Lorenz gauge, $\partial_\mu F^{\mu\nu} = j^\nu$ becomes

$$(\partial_\mu \partial^\mu) A^\nu = j^\nu$$

four-momentum of photon

Trial sol'n

$$A^\mu = N \varepsilon^\mu e^{-ik\cdot x}$$

ε^μ = polarisation (four-) vector (function of k_μ)

In vacuum ($j^\nu = 0$)

$$(\partial_\mu \partial^\mu) A^\nu = -k^2 A^\nu$$

$\Rightarrow$ trial sol'n works if $k^2 = k_\mu k^\mu = 0$

$\rightarrow$ QM excitation of EM field has $E^2 - \vec{p}^{\,2} = m^2 = 0$

($E = i\frac{\partial}{\partial t}$, $\vec{p} = -i\nabla$)

Assume Lorenz gauge, use $A^\mu = N \varepsilon^\mu e^{-ik\cdot x}$,

$$\partial_\mu A^\mu \propto k_\mu \varepsilon^\mu = k^0 \varepsilon^0 - \vec{k}\cdot\vec{\varepsilon} = 0$$

$\rightarrow$ k_μ & ε^μ orthogonal in 4-vector sense

From constraint $k_\mu \varepsilon^\mu = 0$, $\rightarrow$ ε^μ only has 3 indep. components

Lorenz condition ($\partial_\mu A^\mu = 0$) still does not uniquely fix A^μ, can add $A'^\mu = A^\mu + \partial^\mu \lambda$

provided $(\partial_\nu \partial^\nu)\lambda = 0$

($\partial_\mu A'^\mu$ will still be 0).

Suppose in (a particular) Lorenz gauge $\nabla\cdot\vec{A} = g(x)$
$\uparrow$ can make this zero.

Carry out a further gauge trans

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\lambda$$

Choose λ such that $\partial_\mu \partial^\mu \lambda = 0$ so that Lorenz condition $\partial_\mu A'^\mu = \underbrace{\partial_\mu A^\mu}_{\text{was already } 0} + \underbrace{\partial_\mu \partial^\mu \lambda}_{0} = 0$ still holds.

Further, choose λ so that $\nabla^2\lambda = \nabla\cdot\vec{A} = g(x)$

$$\Rightarrow \quad \vec{A}' = \vec{A} - \nabla\lambda \quad \rightarrow \quad \nabla\cdot\vec{A}' = \nabla\cdot\vec{A} - \nabla^2\lambda$$
$$= g(x) - g(x) = \underline{0}$$

$\longrightarrow$ <u>Coulomb gauge</u>

One more constraint $\Rightarrow$ ε^μ only has 2 indep. components

From $\nabla\cdot\vec{A} = -iN\,\vec{k}\cdot\vec{\varepsilon}\,e^{-i\vec{k}\cdot\vec{x}} = 0$

$$\Rightarrow \quad \vec{k}\cdot\vec{\varepsilon} = 0$$

From $k_\mu \varepsilon^\mu = k^0\varepsilon^0 - \vec{k}\cdot\vec{\varepsilon} = 0 \quad \Rightarrow \quad \varepsilon^0 = 0$

$\Rightarrow$ $\vec{\varepsilon}$ $\perp$ to direction of $\vec{k}$

e.g. for $\vec{k}$ along $\hat{z}$ can take polarisation basis vectors $\vec{\varepsilon}_1 = (1,0,0)$, $\vec{\varepsilon}_2 = (0,1,0)$

i.e. $\vec{\varepsilon}_r\cdot\vec{\varepsilon}_s = \delta_{rs}$, $r,s = 1,2$ ~ pol. indices

Recap of non-rel. QM (put back $\hbar$)

Particles associated w/ wave with

$$E = \hbar\omega, \quad \vec{p} = \hbar\vec{k} \qquad \left(|\vec{k}| = \frac{2\pi}{\lambda}\right)$$

① $$\Psi(\vec{r},t) = \frac{1}{\sqrt{V}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

V ← normalising volume $\int_V |\Psi|^2 d^3r = 1$

Observables associated w/ operators

② $$\left.\begin{aligned} \hat{E}\Psi &= E\Psi = \hbar\omega\Psi \\ \hat{\vec{p}}\Psi &= \vec{p}\Psi = \hbar\vec{k}\Psi \end{aligned}\right\}$$ Eigenvalues = possible outcomes of measurement

① + ② $\Rightarrow$ $$\hat{E} = i\hbar\frac{\partial}{\partial t}$$
$$\hat{\vec{p}} = -i\hbar\nabla$$

In non-rel classical mech., for free ptcl

$$E = E_{kin} = \frac{p^2}{2m}$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi$$

Now suppose $E = \frac{p^2}{2m} + V(\vec{r},t)$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\right)\Psi(\vec{r},t)$$

$$\equiv \underbrace{\mathcal{H}}_{\text{Hamiltonian.}}\Psi(\vec{r},t)$$

To interpret Ψ, $|\Psi|^2$ = probability density

$$\left\{\begin{aligned} \mathcal{H}\Psi &= i\hbar\frac{\partial}{\partial t}\Psi \\ \mathcal{H}^*\Psi^* &= -i\hbar\frac{\partial}{\partial t}\Psi^* = \mathcal{H}\Psi^* \end{aligned}\right.$$

$\mathcal{H}^* = \mathcal{H}$ if V real

$$\Psi^*\mathcal{H}\Psi - \Psi\mathcal{H}\Psi^* = i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

use $\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$

$$\Rightarrow \frac{\partial}{\partial t}\left(\Psi^*\Psi\right) + \frac{\hbar}{2im}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

where $\rho = \Psi^*\Psi$ $\quad$ $\nabla\cdot j =$

$$\vec{j} = \frac{\hbar}{2im}\left[\Psi^*\nabla\Psi - (\nabla\Psi)^*\Psi\right]$$

Corresponding electric charge density & current

$$\rho_{e.m.} = q\rho, \qquad \vec{j}_{e.m.} = q\vec{j}$$

charge of ptcl.

For $\Psi(\vec{r},t) = \frac{1}{\sqrt{V}}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\rho = \Psi^*\Psi = \frac{1}{V}$ — for free ptcl., ρ indep. of $\vec{r}$

$$\vec{j} = \frac{\hbar}{2im}\left[i\vec{k}\,\Psi^*\Psi - (-i\vec{k})\,\Psi^*\Psi\right] = \frac{\hbar\vec{k}}{2m}\,2\,\Psi^*\Psi^*$$

$$= \frac{\vec{p}}{m}\rho = \vec{v}\rho$$

The Klein-Gordon eq. (now $\hbar = c = 1$)

Use relativistic relation for $E, \vec{p}, m$

$$E^2 = \vec{p}^{\,2} + m^2$$

and same operators as before

$$\left.\begin{array}{l} \hat{E} = i \dfrac{\partial}{\partial t} \\ \hat{\vec{p}} = -i \nabla \end{array}\right\}$$ operate on wave func. $\varphi(x)$

$\uparrow \vec{r}, t$

$$\Rightarrow \quad -\frac{\partial^2}{\partial t^2} \varphi = \left(-\nabla^2 + m^2\right) \varphi$$

or

$$\left(\partial_\mu \partial^\mu + m^2\right) \varphi(x) = 0$$

Klein-Gordon eq. for free ptcl. of mass m

Solution:

$$\varphi(x) = N e^{\pm i p \cdot x} = N e^{\pm i (Et - \vec{p} \cdot \vec{r})}$$

both signs

set normalisation factor later

Here $E \equiv + \sqrt{|\vec{p}|^2 + m^2} > 0$

and "$p \cdot x$" means $Et - \vec{p} \cdot \vec{r}$

Depending on choice of + or −,

$$\varphi_+ = N e^{-ip\cdot x} \longrightarrow i \frac{\partial}{\partial t} \varphi_+ = + E \varphi_+$$

$$\varphi_- = N e^{+ip\cdot x} \longrightarrow i \frac{\partial}{\partial t} \varphi_- = - E \varphi_-$$

"Energy" $\equiv$ eigenvalue of $i \frac{\partial}{\partial t} = \begin{cases} +E & \text{for } \varphi_+ \\ -E & \text{" } \varphi_- \end{cases}$

What are the neg. energy sol'ns φ_- ?

Cannot ignore (needed to form complete set of sol'ns)

Later use φ_- (in a non-trivial way) to describe antiparticles.

Probability interp. ?

$$\text{Need } \varphi'(x') = \varphi(x) \qquad (\text{Lorentz scalar})$$

$\Rightarrow |\varphi|^2$ not a prob. density

since need $|\varphi|^2 dV$ invariant under boost,

but dV <u>contracted</u> under boost.

Prob. density + current should satisfy a continuity eq. Define

$$j^\mu = (\rho, \vec{j}) \equiv \underbrace{\frac{i}{2m}} \left(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^* \right)$$

make j^μ coincide w/ Schrödinger j^μ in non-rel. limit

Note if φ Lorentz scalar,

$\partial^\mu \varphi \rightarrow j^\mu$ are Lorentz vectors

Can show $\partial_\mu j^\mu = \dfrac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$ (prob. sheet)

Using $\varphi_+ = N e^{-i\overbrace{p\cdot x}^{Et - \vec{p}\cdot\vec{r}}}$, $E > 0$

$$\rho = \frac{i}{2m}\left[(-iE)\varphi_+^*\varphi_+ - (+iE)\varphi_+^*\varphi_+\right] = \frac{E}{m}|N|^2$$

$$\vec{j} = -i\left[(i\vec{p})\varphi_+^*\varphi_+ - (-i\vec{p})\varphi_+^*\varphi_+\right] = \frac{\vec{p}}{m}|N|^2$$

In non-rel limit, $p^0 \to m \Rightarrow \rho \to |N|^2$

so if $N = \frac{1}{\sqrt{V}}$, $\rho \to \frac{1}{V}$ as w/ Schrödinger

For neg energy sol'n φ_-,

$$\rho = i\left[(iE)\varphi_-^*\varphi_- - (-iE)\varphi_-^*\varphi_-\right] = -\frac{E}{m}|N|^2$$

↑ neg. prob??

⇒ Klein, Gordon, Schrödinger give up on K-G eq.

Resurrect later to describe antiparticles

Then more convenient to define normalisation

$$j^\mu = i(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*)$$

Relativity (extra): why j^μ is a four-vec

$$ds^2 = dt^2 - (d\vec{x})^2 \quad \text{is invariant}$$

Proper time τ define through $d\tau = \sqrt{ds^2}$

τ = time measured in particle's rest frame

Four-vector position $x^\mu = (t, \vec{x})$

" " velocity $u^\mu = \dfrac{dx^\mu}{d\tau} = \left(\dfrac{dt}{d\tau}, \dfrac{d\vec{x}}{d\tau} \right)$

$d\tau = \dfrac{dt}{\gamma}$, $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$d\vec{x} = \vec{\beta}\, dt$ (defines $\vec{\beta} = \vec{v}$)

$d\tau^2 = dt^2 - \beta^2 dt^2$

$= dt^2 (1 - \beta^2)$

$\Rightarrow d\tau = dt/\gamma$

$\Rightarrow u^\mu = (\gamma, \vec{\beta}\gamma)$

- u^μ is a four-vector because it is derivative of 4-vec x^μ wrt scalar τ
- $u_\mu u^\mu = \gamma^2 (1 - \vec{\beta}\cdot\vec{\beta}) = 1 = \text{const.}$

Four-momentum $p^\mu = m u^\mu = (\gamma m, \vec{\beta}\gamma m)$

$\Rightarrow p_\mu p^\mu = m^2$

For particle of charge q, $j^\mu(x) = q u^\mu \delta^{(3)}(\vec{x} - \vec{x}(t))$

or for charge density ρ_0 $j^\mu(x) = \rho_0 u^\mu$

$\Rightarrow j^\mu = (\rho, \vec{j})$ is a four vector.