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PH4442 Advanced Particle Physics Week 2

[Finish review of Schrödinger & Klein-Gordon eqs. (Week 1 notes + slides)]

Dirac Eq.

K-G eq. had neg. energy sol'ns because of

$$E^2 \rightarrow -\frac{\partial^2}{\partial t^2} \rightarrow E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

$\Rightarrow$ Look for eq. w/ only $\frac{\partial}{\partial t}$

Lorentz covariance $\Rightarrow$ also only $\frac{\partial}{\partial x}$

$\Big\}$ $\rightarrow$ Dirac eq.

Turns out, Dirac eq. also has $E<0$ sol'ns, can reinterpret these to give antiparticles

Ansatz for Dirac eq.

some constants

(1) $$i\frac{\partial \psi}{\partial t} = -i\left(\alpha_1 \frac{\partial \psi}{\partial x_1} + \alpha_2 \frac{\partial \psi}{\partial x_2} + \alpha_3 \frac{\partial \psi}{\partial x_3}\right) + \beta m \psi$$

$x, y, z \rightarrow x_1, x_2, x_3$

Retain $\hat{E} = i\frac{\partial}{\partial t}$, $\hat{p} = -i\nabla$ $\Rightarrow$ $H = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m$

& still want $E^2 = |\vec{p}|^2 + m^2$

Apply to (1) $i\frac{\partial}{\partial t}$ on left, $H = \vec{\alpha}\cdot(-i\nabla) + \beta m$ on right

& retain order of all α_i, β

(since will find later they must be matrices & do not commute)

(2) $$-\frac{\partial^2\psi}{\partial t^2} = -\sum_{i=1}^{3} \alpha_i^2 \frac{\partial^2\psi}{\partial x_i^2} + m^2\beta^2\psi$$
$$- \sum_{i\neq j} \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2\psi}{\partial x_i \partial x_j} - im\sum_{i=1}^{3}(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x_i}$$

For a free particle want plane-wave sol'ns

(3) $$\psi \sim e^{\pm i p\cdot x} = e^{\pm i(Et - \vec{p}\cdot\vec{x})}$$

Try to choose α_i, β such that $E^2 = |\vec{p}|^2 + m^2$

If we take

$$\left.\begin{aligned} &\alpha_i^2 = \beta^2 = 1, && i = 1,2,3 \\ &\alpha_i\alpha_j + \alpha_j\alpha_i = 0 && i \neq j \\ &\alpha_i\beta + \beta\alpha_i = 0 && i = 1,2,3 \end{aligned}\right\} \quad (4)$$

then (2), (3), (4) give

$$E^2 = -\sum_{i=1}^{3}(i p_i)^2 + m^2 + 0 - im\,0$$

$$\Rightarrow \quad E^2 = |\vec{p}|^2 + m^2 \quad \checkmark$$

But real or complex α_i, β cannot fulfill (4)

$\Rightarrow$ Look for complex matrices.

Need H Hermitian ($H = H^{\dagger}$) so that time evolution op. $U = e^{-iHt}$ is unitary.

$$H = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m = -i\vec{\alpha} \cdot \nabla + \beta m$$

Hermitian

$\Rightarrow \vec{\alpha}, \beta$ must also be Hermitian ($\Rightarrow$ eigenvalues real)

From $\alpha_i^2 = \beta^2 = I$ (unit matrix), eigenvalues of α_i, β are ± 1

$$\left(\text{e.g. } \underbrace{L^2}_{=I} \psi = \psi, \quad L\psi = \lambda\psi \quad \Rightarrow \underbrace{L^2}\psi = \lambda L \psi = \lambda^2 \psi = \psi \quad \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1 \right)$$

& can show α_i, β all traceless, e.g.,

use $\beta^2 = I$, $\beta\alpha_i = -\alpha_i\beta$

$$\begin{aligned} \mathrm{Tr}(\alpha_i) &= \mathrm{Tr}(\beta^2 \alpha_i) \\ &= \mathrm{Tr}(-\beta\alpha_i\beta) \\ &= -\mathrm{Tr}(\alpha_i\beta^2) \quad \leftarrow \text{trace of product invariant under cyclic permutation} \\ &= -\mathrm{Tr}(\alpha_i) \end{aligned}$$

$\Rightarrow \mathrm{Tr}(\alpha_i) = 0$

& similarly $\mathrm{Tr}(\beta) = 0$.

Next, use trace of matrix = sum of eigenvalues

Since eigenvalues of α_i, β are ± 1,

their dimension must be <u>even</u>.

For $N = 2$, only 3 indep. 2×2 Hermitian matrices (e.g. Pauli)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

but we need 4: $\alpha_1, \alpha_2, \alpha_3, \beta$

$\Rightarrow$ smallest dimension is $N = 4$, e.g.

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Summary: Dirac eq.

$$i \frac{\partial \psi}{\partial t} = -i \sum_{i=1}^{3} \alpha_i \frac{\partial \psi}{\partial x_i} + \beta m \psi$$

$$\equiv H_D \psi, \quad H_D = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$$

$$= -i \vec{\alpha} \cdot \nabla + \beta m$$

where wave func. ψ is 4-component <u>Dirac spinor</u>

$\vec{\alpha}$: $(\alpha_1, \alpha_2, \alpha_3)$ vector of matrices

column vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

⚠ 4 components but <u>not</u> Lorentz 4-vec.

Label e.g. ψ_a, $a = 1, \ldots, 4$

complex conj. & transpose

Hermitian conjugate eq.

$$-i\frac{\partial\psi^\dagger}{\partial t} = i\underbrace{\sum_{i=1}^{3}\frac{\partial\psi^\dagger}{\partial x_i}\alpha_i}_{} + m\underbrace{\psi^\dagger\beta}_{}$$

note order

where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ ← row vector

Continuity eq.: take $\psi^\dagger \cdot$ (Dirac eq. for ψ) $-$ (Dirac eq. for $\psi^\dagger$) $\cdot\,\psi$

$$i\psi^\dagger\frac{\partial\psi}{\partial t} - \left(-i\frac{\partial\psi^\dagger}{\partial t}\cdot\psi\right)$$

$$= -i\left(\sum_i \psi^\dagger\alpha_i\frac{\partial\psi}{\partial x_i}\right) + \cancel{m\psi^\dagger\beta\psi}$$

$$-\left[i\left(\sum_i\frac{\partial\psi^\dagger}{\partial x_i}\alpha_i\psi\right) + \cancel{m\psi^\dagger\beta\psi}\right]$$

$$\Rightarrow i\left[\psi^\dagger\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\psi\right] = -i\left[\sum_i\left(\psi^\dagger\alpha_i\frac{\partial\psi}{\partial x_i} + \frac{\partial\psi^\dagger}{\partial x_i}\alpha_i\psi\right)\right]$$

$$\Rightarrow i\frac{\partial}{\partial t}(\psi^\dagger\psi) = -i\sum_i\frac{\partial}{\partial x_i}(\psi^\dagger\alpha_i\psi)$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

with $\rho = \psi^\dagger\psi = \sum_{a=1}^{4}|\psi_a|^2 > 0$

$$\underbrace{\vec{j} = \psi^\dagger\vec{\alpha}\,\psi}_{}$$

$$\hookrightarrow j^\mu = (\rho, \vec{j})$$

Non-relativistic limit of Dirac eq.

Consider electron at rest: $\vec{p} = 0 \Rightarrow \nabla\psi = 0$

Dirac eq. becomes $i\dfrac{\partial\psi}{\partial t} = m\beta\psi$, i.e.,

$$i\begin{pmatrix} \frac{\partial\psi_1}{\partial t} \\ \frac{\partial\psi_2}{\partial t} \\ \frac{\partial\psi_3}{\partial t} \\ \frac{\partial\psi_4}{\partial t} \end{pmatrix} = m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

4 linearly indep. solutions

2 have energy $\underset{\nwarrow\ \text{eigenvalue of } i\frac{\partial}{\partial t}}{E} > 0$:

$$\psi^{(1)} = e^{-imt}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(2)} = e^{-imt}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_1 = E_2 = m$$

2 sol'ns have $E < 0$

$$\psi^{(3)} = e^{imt}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^{(4)} = e^{imt}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_3 = E_4 = -m.$$

Consider sol'n for $E = \sqrt{p^2+m^2} \approx m + \frac{p^2}{2m}$

$p \ll m$

Time dep. still approx $\sim e^{-imt}$

Ansatz: $$\psi(\vec{r},t) = e^{-imt}\begin{pmatrix}\varphi(\vec{r},t)\\ \chi(\vec{r},t)\end{pmatrix} \qquad (1)$$

φ, χ two-component spinors w/ "slow" time dep.

$$\varphi, \chi \sim e^{-i\omega t}, \qquad \omega = \frac{p^2}{2m} \ll m$$

$$(\text{so } \psi \sim e^{-iEt}, \quad E = m+\omega)$$

Insert (1) into Dirac eq. $i\frac{\partial\psi}{\partial t} = -i\sum_i \alpha_i \frac{\partial\psi}{\partial x_i} + \beta m\psi$

$$\Rightarrow i\left[e^{-imt}\begin{pmatrix}\dot\varphi\\ \dot\chi\end{pmatrix} + (-im)e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\right]$$

$$= -i e^{-imt}\sum_i \alpha_i \frac{\partial}{\partial x_i}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + \beta m e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

$$i\begin{pmatrix}\dot\varphi\\ \dot\chi\end{pmatrix} - im\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = -i\sum_i\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}\begin{pmatrix}\partial\varphi/\partial x_i\\ \partial\chi/\partial x_i\end{pmatrix} + m\begin{pmatrix}I_2 & 0\\ 0 & I_2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

Now use $p_i = -i\frac{\partial}{\partial x_i}$ (drop hats)

$$\Rightarrow m\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + i\begin{pmatrix}\dot\varphi\\ \dot\chi\end{pmatrix} = \underbrace{\vec\sigma\cdot\vec p}\begin{pmatrix}\chi\\ \varphi\end{pmatrix} + m\begin{pmatrix}\varphi\\ -\chi\end{pmatrix} \qquad (2)$$

$$\begin{pmatrix}\vec\sigma\cdot\vec p & 0\\ 0 & \vec\sigma\cdot\vec p\end{pmatrix}$$

Lower eq. is

$$m\chi + i\dot{\chi} = \vec{\sigma}\cdot\vec{p}\,\varphi - m\chi$$

Try Ansatz $\varphi, \chi \sim e^{-i\omega t}$ w/ $\omega = \frac{p^2}{2m}$

$$\Rightarrow \quad i\dot{\chi} = \omega\chi = \vec{\sigma}\cdot\vec{p}\,\varphi - 2m\chi$$

But since $\omega \ll m$,

$$\boxed{\chi \simeq \frac{1}{2m}(\vec{\sigma}\cdot\vec{p})\varphi} \qquad (3)$$

Use relations

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}$$

$$\Rightarrow \quad (\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \chi^\dagger\chi = \frac{1}{4m^2}\varphi^\dagger(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})\varphi = \frac{p^2}{4m^2}\varphi^\dagger\varphi$$

$$\Rightarrow \quad \chi^\dagger\chi \ll \varphi^\dagger\varphi \quad \text{for} \quad \frac{p^2}{2m} \ll m \qquad \left(= \frac{p^2/2m}{2m}\varphi^\dagger\varphi \right)$$

φ = large component

χ = small "

For neg energy sol'ns, roles of φ, χ switch. In non-rel limit, neg.-E sol'ns decouple since separated by gap $\Delta E = 2m$

For large component φ of (2),

$$i\dot{\varphi} = (\vec{\sigma}\cdot\vec{p})\chi = \underbrace{\frac{(\vec{\sigma}\cdot\vec{p})^2}{2m}}_{\text{from } (3)}\varphi$$

$$\Rightarrow i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m^2}\nabla^2\varphi$$ $\Rightarrow$ ~ Schrödinger, but φ has two components

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Dirac eq. for e^- in electromag. field

For EM field w/ potentials $\phi, \vec{A}$, charge q

classical recipe: $V \rightarrow V + q\phi$ "minimal substitution" in
(see, e.g., Goldstein) $\vec{p} \rightarrow \vec{p} - q\vec{A}$ classical Hamiltonian

Start from free-ptcl Dirac Hamiltonian

$$\mathcal{H} = \vec{\alpha}\cdot\vec{p} + m\beta \qquad \left(\begin{array}{l}\vec{p} = -i\nabla \\ V = 0\end{array}\right)$$

Make minimal sub:

$$i\frac{\partial\Psi}{\partial t} = \left[\vec{\alpha}\cdot(\vec{p} - q\vec{A}) + m\beta + q\phi\right]\Psi$$

For large component

$$i\dot{\varphi} = \frac{1}{2m}\left[\vec{\sigma}\cdot(\vec{p} - q\vec{A})\right]^2\varphi + q\phi\varphi$$

$$= \frac{1}{2m}\left[\vec{\sigma}\cdot(\underbrace{-i\nabla}_{\text{acts on } \vec{A}(\vec{x}),\ \varphi(\vec{x})} - q\vec{A})\right]\left[\vec{\sigma}\cdot(-i\nabla - q\vec{A})\right]\varphi + q\phi\varphi$$

... long calc ... produces terms like $\nabla\times\vec{A} = \vec{B}$ (!)

e.g. terms like

$$\sigma_x\sigma_y\left(-i\frac{\partial}{\partial x} - qA_x\right)\left(-i\frac{\partial}{\partial y} - qA_y\right) + \sigma_x\sigma_y\left(-\frac{\partial}{\partial y} - qA_y\right)\left(-i\frac{\partial}{\partial x} - qA_x\right)$$

For e^- w/ $q = -e$, find

$$\boxed{i\frac{\partial \psi}{\partial t} = \Big[\frac{1}{2m}(-i\nabla + e\vec{A})^2 - e\phi + \underbrace{\frac{e}{2m}\vec{\sigma}\cdot\vec{B}}_{H_B = \frac{e}{2m}\vec{\sigma}\cdot\vec{B}}\Big]\psi}$$

Pauli equation

$H_B = \frac{e}{2m}\vec{\sigma}\cdot\vec{B}$ = potential energy from electron's magnetic moment in $\vec{B}$ field

Compare to Schrödinger:

$$i\frac{\partial \psi_s}{\partial t} = \Big[\frac{1}{2m}(-i\nabla + e\vec{A})^2 - e\phi\Big]\psi_s$$

~ same, except ψ_s has single component (no spin)

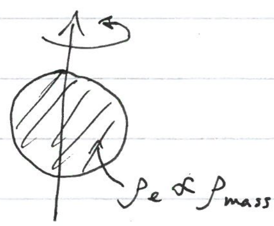
and no mag. moment term $\frac{e}{2m}\vec{\sigma}\cdot\vec{B}$

Spin $\vec{S} = \frac{1}{2}\vec{\sigma}$ + 2-component wave func. already introduced

Uhlenbeck + Goudsmit 1925

Classically, for body of mass m, charge $-e$

mag. moment is $\vec{\mu} = -\frac{e}{2m}\vec{L}$ ← orb. ∡ mom.



pot. energy $V_B = -\vec{\mu}\cdot\vec{B}$

But e^- not classical! Don't expect $\vec{L}\to\vec{S}$, add $-\vec{\mu}\cdot\vec{B}$ to Hamiltonian

Guess: $H_B = -\vec{\mu}_e\cdot\vec{B}$

$$\vec{\mu}_e = -g\frac{e}{2m}\vec{S} = -g\frac{e}{2m}\frac{1}{2}\vec{\sigma}$$

⇒ Dirac <u>predicts</u> $g = 2$

$g_{exp} = 2.00231930436118 \pm 0.00000000000026$

} agrees to $\sim 10^{-3}$. With higher-order corrections, ~perfect agreement

⇒ QM + relativity → behaviour of charge in magnetic field

Covariant form of Dirac eq.

Define Dirac gamma matrices

upper index
$$\begin{cases} \gamma^0 = \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \\ \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3 \end{cases}$$

Lower index versions as usual $\gamma_\mu = g_{\mu\nu}\gamma^\nu$

$[\ \gamma_0 = \gamma^0, \quad \gamma_i = -\gamma^i\ ]$ $\qquad \mu, \nu \in \{0,1,2,3\}$ (Lorentz indices)

But, γ^μ not a four-vector,

has same form in all reference frames

Can show ($\rightarrow$ prob. sheet)

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} I_4$$

4x4 unit matrix

μ,ν element of g

$i=1,2,3$

$$\Rightarrow (\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1$$

1 means I_4

For particle of mass m, charge q in $(\phi, \vec{A})$

$$i\left(\frac{\partial\psi}{\partial t} + \vec{\alpha}\cdot\nabla\psi\right) - m\beta\psi = q(\vec{\alpha}\cdot\vec{A} - \phi)\psi$$

Mult on left by γ^0 + use $(\gamma^0)^2 = 1 \quad (= I_4)$

$$\left[i\left(\gamma^0\frac{\partial}{\partial x^0} + \gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3}\right) - m I_4\right]\psi$$

$$= -q(\gamma^0\phi - \gamma^1 A^1 - \gamma^2 A^2 - \gamma^3 A^3)\psi$$

note + for $\frac{\partial}{\partial x^i}$ terms

Use four-vector potential $A^\mu = (\phi, \vec{A})$

$(A_\mu = (\phi, -\vec{A}))$

$$\Rightarrow \quad (i\gamma^\mu \underbrace{\partial_\mu}_{\to\left(\frac{\partial}{\partial t},\, \nabla\right)} - m)\psi = q\gamma^\mu A_\mu$$

Feynman "slash" notation

$$\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu \quad \leftarrow \text{any four-vector}$$

$$\Rightarrow \quad \boxed{(i\not{\partial} - m)\psi = q\not{A}\psi} \qquad \begin{array}{l} \not{\partial} = \gamma^\mu \partial_\mu \\ \not{A} = \gamma^\mu A_\mu \end{array}$$

Properties of gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4 \qquad \text{(Clifford algebra)}$$

Contracted combinations of gamma matrices, e.g.

$$\gamma^\mu \gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu$$

$$= g_{\mu\nu}\left[\frac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{\text{symmetric } \mu \leftrightarrow \nu} + \frac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)}_{\text{antisymmetric in } \mu \leftrightarrow \nu}\right]$$

If $S_{\mu\nu}$ symmetric, $a^{\mu\nu}$ antisymmetric,

$$\Rightarrow \quad S_{\mu\nu}a^{\mu\nu} = -S_{\nu\mu}a^{\nu\mu} = \underbrace{-S_{\mu\nu}a^{\mu\nu}}_{\text{relabel } \mu \leftrightarrow \nu} = \underline{\underline{0}}$$

} useful general result

$$\Rightarrow \gamma^\mu \gamma_\mu = g_{\mu\nu} \left[\frac{1}{2} \underbrace{(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)}_{2 g^{\mu\nu} I_4} + \frac{1}{2} \cancelto{0}{(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)} \right]$$

$$= \underbrace{g_{\mu\nu} g^{\mu\nu}}_{\delta_\mu^\mu = 4} I_4 = \underline{4 I_4}$$

Often encounter adjoint $\gamma^{\mu +} = (\gamma^\mu)^{*T}$

One finds $\gamma^{0+} = \gamma^0$

$\gamma^{i+} = -\gamma^i \qquad i = 1, 2, 3$

or equivalently $\gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0$

$(\longrightarrow \gamma^\mu = \gamma^0 \gamma^{\mu +} \gamma^0)$

Some more (w/o proof)

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho} I_4$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2 \gamma^\sigma \gamma^\rho \gamma^\nu$$

For weak interactions we will use

$$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

Properties: $(\gamma^5)^+ = \gamma^5$

$(\gamma^5)^2 = I_4$

$\{\gamma^5, \gamma^\mu\} = 0$

Solving Dirac eq. for free ptcl.

$$(i\not{\partial} - m)\psi(x) = 0$$

Prelude: show that all 4 components of ψ individually satisfy Klein-Gordon

$$\underbrace{(i\not{\partial}+m)}_{\text{apply on left}}(i\not{\partial}-m)\psi = 0$$

$$\Rightarrow (\not{\partial}\not{\partial} + m^2)\psi = (\gamma^\mu\gamma^\nu \underbrace{\partial_\mu\partial_\nu}_{\text{symmetric}} + m^2)\psi = 0$$

$$\gamma^\mu\gamma^\nu = \underbrace{\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{\text{sym.}} + \underbrace{\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]}_{\text{antisym.}}$$

antisym. $\Rightarrow$ gives 0 when contract w/ $\partial_\mu\partial_\nu$

$$\Rightarrow \Big[\tfrac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}I_4}\partial_\mu\partial_\nu + m^2\Big]\psi$$

$$= (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)I_4\psi = (\partial_\mu\partial^\mu + m^2)I_4\psi = 0$$

i.e. $(\partial_\mu\partial^\mu + m^2)\psi_a = 0$, $\quad a = 1, \dots, 4$

Dirac spinors u and v ($\rightarrow$ slides)

Each component of ψ solves K-G

$\Rightarrow$ sol'ns have form:

$$\psi_+(x) = \underset{\text{4 components}}{u(p)}\, e^{-ip\cdot x} \quad \text{or pos. energy } p^0 = E$$

$$\psi_-(x) = v(p)\, e^{ip\cdot x} \quad \text{or neg. energy } p^0 = -E$$

where energy p^0 = eigenvalue of $i\frac{\partial}{\partial t}$

$$E \equiv \sqrt{|\vec{p}|^2 + m^2} > 0$$

here $\quad p\cdot x = Et - \vec{p}\cdot\vec{x} \quad$ (not $p^0 t - \vec{p}\cdot\vec{x}$)

$\Rightarrow \quad \psi_+ \sim e^{-iEt}$ has "normal" time dep.

$\psi_- \sim e^{+iEt}$ does <u>not</u> (& does not solve Schrödinger)

For ψ_+ in Dirac eq.

$$(i\partial\!\!\!/ - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$(i(-ip\!\!\!/) - m)\, u(p)\, \cancel{e^{-ip\cdot x}} = 0$$

$$\Rightarrow \quad (p\!\!\!/ - m)\, u(p) = 0$$

+ similarly $\quad (p\!\!\!/ + m)\, v(p) = 0$

Define (Dirac) adjoint spinors $\bar{u} = u^\dagger \gamma^0$, $\bar{v} = v^\dagger \gamma^0$ ← row vectors

find: $\quad \bar{u}\,(p\!\!\!/ - m) = 0$

$\quad\quad\quad \bar{v}\,(p\!\!\!/ + m) = 0$

Rest-frame solutions

$$p^\mu = (p^0, 0, 0, 0) = (m, \vec{0})$$

Use $\not{p} = \gamma^\mu p_\mu = \gamma^0 m$ & cancel m

$$\Rightarrow \quad (\gamma^0 - I)\, u(m, \vec{0}) = 0 \qquad (1)$$

$$(\gamma^0 + I)\, v(m, \vec{0}) = 0 \qquad (2)$$

Use $\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (Dirac representation)

(1) becomes $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$

Solved for arbitrary a, b with $c = d = 0$

$\Rightarrow$ 2 linearly indep. solutions are

$$u_s(m, \vec{0}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = N \begin{pmatrix} \varphi_s \\ \underbrace{0}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \end{pmatrix}, \quad \underset{\text{"spin" index}}{\underset{\uparrow}{s = 1, 2}}$$

Can choose $\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ — "Weyl spinors"

Similarly $v_s(m, \vec{0}) = N \begin{pmatrix} 0 \\ \chi_s \end{pmatrix}$, $\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

choose so v_1 spin up relative to $\hat{z}$

i.e. $\varphi_r^\dagger \varphi_s = \chi_r^\dagger \chi_s = \delta_{rs}$

Note this does not yet fix normalisation of u_s, v_s.

Solutions for four-momentum p

Can obtain from rest-frame sol'ns:

$$u_s(p) = (\not{p} + m)\, u_s(m, \vec{0})$$

$$v_s(p) = (\not{p} - m)\, v_s(m, 0)$$

Proof: $(\not{p} - m)\, u_s(p) = (\not{p} - m)(\not{p} + m)\, u_s(m, \vec{0})$

$$= (\not{p}\not{p} - m^2)\, u_s(m, \vec{0})$$

$$= 0 \quad \checkmark$$

because $\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = p^2 = m^2$

$\hookrightarrow$ use symmetry + $\{\gamma^\mu, \gamma^\nu\}$

For explicit form of sol'ns, recall

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\Rightarrow \quad \not{p} = \gamma^\mu p_\mu = \begin{pmatrix} p^0 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p^0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} u_s(p) = \begin{pmatrix} p^0+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p^0+m \end{pmatrix}\begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} = \begin{pmatrix} (p^0+m)\,\varphi_s \\ (\vec{\sigma}\cdot\vec{p})\,\varphi_s \end{pmatrix} \\ v_s(p) = \begin{pmatrix} p^0-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -p^0+m \end{pmatrix}\begin{pmatrix} 0 \\ \chi_s \end{pmatrix} = \begin{pmatrix} (\vec{\sigma}\cdot\vec{p})\,\chi_s \\ (p^0+m)\,\chi_s \end{pmatrix} \end{cases}$$

not yet normalised

Normalisation of spinors

Recall $\rho = \psi^\dagger\psi$ = prob. density

$= 0^{th}$ component of $j^\mu = (\rho, \vec{j})$

Prob (ptcl. in vol. V) should be Lorentz invariant
(also transforms)

$$\doteq \int_V d^3x \; j^0(x) = \int d^3x \; \psi^\dagger(x)\,\psi(x)$$

i.e. ρ

$$= \begin{cases} \int d^3x \; u^\dagger u & \text{for } \psi_+ \\ \int d^3x \; v^\dagger v & \text{for } \psi_- \end{cases} \qquad \left(\text{exp terms } |e^{\pm ip\cdot x}|^2 = 1\right)$$

Will be Lorentz invar. if $\rho = \psi^\dagger\psi$ transforms like 0^{th} component of four-vector

Only four-vector on which this can depend is $\underline{p^\mu}$

$\Rightarrow$ take $u^\dagger u \propto v^\dagger v \propto p^0$

[See Peskin: $\rho \to \rho\gamma$, $dV \to dV/\gamma$ $\Rightarrow$ $\rho\, dV$ invariant

Take convention

$$u_r^\dagger u_s = v_r^\dagger v_s = 2E\,\delta_{rs}$$

$\sqrt{|\vec{p}|^2+m^2} > 0$

[some texts use $u_r^\dagger u_s = v_r^\dagger v_s = \frac{E}{m}\,\delta_{rs}$]

From this can show for $\bar{u} = u^\dagger \gamma^0$, $\bar{v} = v^\dagger \gamma^0$

$$\bar{u}_r u_s = 2m\,\delta_{rs}$$

$$\bar{v}_r v_s = -2m\,\delta_{rs}$$

(also $\bar{u}v = \bar{v}u = 0$)

From this normalisation finally find

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$\Uparrow$ $\qquad\qquad$ $\Downarrow$

spin relative to z axis

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

$\Uparrow$ $\qquad\qquad$ $\Downarrow$

$E>0$ & $\vec{p}$ are physical energy & momentum

Full sol'ns:

$$\psi_+(x) = u_s(p)\,e^{-ip\cdot x}$$

$$\psi_-(x) = v_s(p)\,e^{ip\cdot x} \qquad s = 1,2$$

(drop factors $\frac{1}{\sqrt{V}}$, i.e., set $V=1$)