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PH4442 Advanced Particle Physics Week 3

Recall from last week:

Dirac eq. $$(i\not\partial - m)\psi = \begin{cases} q\not{A}\psi & \text{in EM pot.} \\ 0 & \text{free ptcl} \end{cases}$$

Free ptcl sol'ns $\psi_+ = u_s(p)\, e^{-ip\cdot x}$

see video + finish today $\psi_- = \underbrace{v_s(p)}_{\text{Dirac spinors}}\, e^{ip\cdot x}$

Today:

0) Finish discussion of u, v spinors

1) Transformation properties / covariance of K-G + Dirac eqs.

2) Interpretation of negative energy sol'ns → antiparticles

3) Green functions (propagators) } start today

Where this is heading:

- Feynman rules for amplitudes
- Predictions for decay rates and cross sections.

[Covariance of Maxwell's eqs. ← see notes]

Covariance of Klein - Gordon eq.

Consider two reference frames related by $x' = \Lambda x$

Lorentz trans

In original frame, K-G eq. is

$$(\partial_\mu \partial^\mu + m^2)\,\varphi(x) = 0$$

We <u>assume</u> $\varphi(x)$ is a Lorentz scalar, i.e.,

in primed frame: $\varphi'(x') = \varphi(x)$

We showed (prob sheet 1)

$$\partial'_\mu \partial'^\mu \varphi'(x') = \partial_\mu \partial^\mu \varphi(x)$$

⇒ K-G eq. in primed frame:

$$(\partial'_\mu \partial'^\mu + m^2)\,\varphi'(x') = 0$$

⇒ 1) Eqs. same form in both frames

2) We have a rule that relates

$\varphi'(x') \longleftrightarrow \varphi(x)$ (here, they're equal)

⇒ K-G eq. is Lorentz <u>covariant</u>

Covariance of Dirac eq.

want this $\rightarrow$ $\begin{cases} \text{Original frame}: & (i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0 \quad (1) \\ \text{Primed frame}: & (i\gamma^\mu \partial'_\mu - m)\,\psi'(x') = 0 \quad (2) \end{cases}$

$\uparrow$ note no prime for γ^μ

The γ^μ do not transform, rather, they specify the coupled system of DEs (same in all frames)

For $x' = \Lambda x$ and assuming Dirac eq. has same form, what is relation between $\psi'(x')$ and $\psi(x)$?

Try $\psi'(x') = \underbrace{S(\Lambda)}\,\psi(x)$

$\uparrow$ 4×4 matrix

$$\Rightarrow \psi(x) = S^{-1}(\Lambda)\,\psi'(x')$$

Use $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \dfrac{\partial x'^\nu}{\partial x^\mu}\dfrac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu\,\partial'_\nu$

use in Dirac eq (1)

$$\rightarrow \left(i\gamma^\mu \Lambda^\mu{}_\nu\,\partial'_\nu - m\right) S^{-1}(\Lambda)\,\psi'(x') = 0$$

Multiply on left by $S(\Lambda)$, and use

$$\gamma^\mu \Lambda^\nu{}_\mu = \underset{\text{number}}{\Lambda^\nu{}_\mu} \; \underset{4\times 4 \text{ matrix}}{\gamma^\mu}$$

$$\Rightarrow \quad (i S \Lambda^\nu{}_\mu \gamma^\mu S^{-1} \partial'_\nu - m)\psi'(x') = 0$$

i.e. same form, $(i\gamma^\nu \partial'_\nu - m)\psi'(x') = 0$

if $\underline{S \Lambda^\nu{}_\mu \gamma^\mu S^{-1} = \gamma^\nu}$.

Optional: Lecture Notes Sec. 2.6

Exponential form of Lorentz trans. (continuous)

$$\Lambda = \exp\left(-\tfrac{1}{2}\omega_{\alpha\beta} L^{\alpha\beta}\right) \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}, \quad L^{\alpha\beta} = -L^{\beta\alpha}$$

"generators" of Lorentz trans.

$\omega_{12} = \theta_3, \; \omega_{13} = -\theta_2, \; \omega_{23} = \theta_1, \; \Big\}$ rotations

$\omega_{01} = \zeta_1, \; \omega_{02} = \zeta_2, \; \omega_{03} = \zeta_3 \; \Big\}$ boosts w/ rapidity $\zeta = \cosh^{-1}\gamma$

What is corresponding $S(\Lambda)$? Claim:

same

$$S = \exp\left(-\tfrac{1}{2}\omega_{\alpha\beta}\Sigma_1^{\alpha\beta}\right)$$

$$\Sigma_1^{\alpha\beta} = \tfrac{1}{4}[\gamma^\alpha, \gamma^\beta]$$

Proof in Lecture Notes Sec. 5.3

Sections 2.6 and start of 5.3 <u>not</u> examinable (Examinable from 5.3.1)

Rotation of coordinates

What Λ does this correspond to?

Consider $S = \exp\left(-\frac{\theta}{2}\gamma^1\gamma^2\right) = I - \frac{\theta}{2}\gamma^1\gamma^2 + \frac{1}{2!}\left(\frac{\theta}{2}\right)^2(\gamma^1\gamma^2)^2$

$+ \ldots$

From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \Rightarrow \gamma^1\gamma^2 = -\gamma^2\gamma^1$

$$\Rightarrow (\gamma^1\gamma^2)^2 = \gamma^1\underbrace{\gamma^2\gamma^1}_{-\gamma^1\gamma^2}\gamma^2 = -\underbrace{\gamma^1\gamma^1}_{-I}\ \underbrace{\gamma^2\gamma^2}_{-I} = -I$$

i.e. $\gamma^1\gamma^2$ like imaginary unit "i", $i^2 = -1$

$$\Rightarrow S = I\cos\frac{\theta}{2} - \gamma^1\gamma^2\sin\frac{\theta}{2}$$

$$S^{-1} = I\cos\frac{\theta}{2} + \gamma^1\gamma^2\sin\frac{\theta}{2}$$

can make rigorous by expanding exp, sin, cos in Taylor series.

Use $S\Lambda^\nu{}_\mu\gamma^\mu S^{-1} = \gamma^\nu$, mult on left by S^{-1}, on right by S

$$\Rightarrow \underline{S^{-1}\gamma^\nu S = \Lambda^\nu{}_\mu\gamma^\mu}$$ $\rightsquigarrow$ 4 eqs., use to find $\Lambda^\nu{}_\mu$

$$S^{-1}\gamma^0 S = \gamma^0\cos^2\frac{\theta}{2} - \underbrace{\gamma^1\gamma^2\gamma^0\gamma^1\gamma^2}_{+\gamma^0}\sin^2\frac{\theta}{2} = \gamma^0$$

$$S^{-1}\gamma^1 S = \left(I\cos\frac{\theta}{2} + \gamma^1\gamma^2\sin\frac{\theta}{2}\right)\gamma^1\left(\cos\frac{\theta}{2} - \gamma^1\gamma^2\sin\frac{\theta}{2}\right)$$

$$= \gamma^1\cos\theta + \gamma^2\sin\theta$$ $\rightsquigarrow$ used $\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{1}{2}\cos\theta$

$\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$

Similarly, $S^{-1}\gamma^2 S = -\gamma^1\sin\theta + \gamma^3\cos\theta$

$S^{-1}\gamma^3 S = \gamma^3$

$$\Rightarrow \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ $\rightsquigarrow$ rotation about z axis.

$$\Rightarrow S_R(\theta) = \exp\left(-\frac{\theta}{2}\gamma^1\gamma^2\right) \qquad = I\cos\frac{\theta}{2} - \gamma^1\gamma^2\sin\frac{\theta}{2}$$

rotation

Inverse $S_R^{-1}(\theta) = \exp\left(+\frac{\theta}{2}\gamma^1\gamma^2\right)$

i.e. $S_R^{-1}(\theta) = S_R(-\theta)$

Note that
$$\left.\begin{aligned} S_R(2\pi) &= -I \\ S_R(4\pi) &= +I \end{aligned}\right\}$$

ψ comes back to itself only after $\underline{2}$ full rotations (Observable consequences in interference phenom., e.g., neutron scattering.)

<u>Show Dirac ptcls have spin $\frac{1}{2}$</u>

Consider infinitesimal rot. $\delta\theta$ about x^3, $\quad x = \Lambda^{-1}x'$

$\theta \to -\theta$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\delta\theta & -\sin\delta\theta & 0 \\ 0 & \sin\delta\theta & \cos\delta\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\theta & 0 \\ 0 & \delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

$$S_R(\delta\theta) = \exp\left(-\frac{\delta\theta}{2}\gamma^1\gamma^2\right) \approx I - \frac{\delta\theta}{2}\underbrace{\gamma^1\gamma^2}$$

check this!

$$= I + i\frac{\delta\theta}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

Pauli matrix

$$\Rightarrow \psi'(x') = \left[I + i\frac{\delta\theta}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right]\psi(x) + O(\delta\theta^2)$$

Express $\psi(x)$ in terms of transformed coords. & expand to first order about x'.

$$\psi(x) = \psi(x'^0,\ x'^1 - x'^2\,\delta\theta,\ x'^1\,\delta\theta + x'^2,\ x'^3)$$

$$\simeq \psi(x') + \left(x'^1 \frac{\partial \psi}{\partial x'^2} - x'^2 \frac{\partial \psi}{\partial x'^1} \right)\delta\theta \ + \ O(\delta\theta^2)$$

$$\Rightarrow \psi'(x') = \left[I + i\frac{\delta\theta}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \right] \left(\psi(x') + \left(x'^1 \frac{\partial \psi}{\partial x'^2} - x'^2 \frac{\partial \psi}{\partial x'^1} \right) \right) + O(\delta\theta^2)$$

Holds for all coord. values $\Rightarrow$ relabel $x' \to x$

$$\Rightarrow \psi'(x) = \left[I + i\,\delta\theta\,(L_3 + S_3) \right] \psi(x)$$

where $L_3 = -i\left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \right)$ = 3rd component of $\vec{L} = \vec{r} \times \vec{p}$ $\quad -i\nabla$

S_3 = 3rd component of $\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

For finite rotation θ, let $\delta\theta = \frac{\theta}{N}$ and apply infinitesimal rot. N times, let $N \to \infty$:

$$\psi'(x) = \lim_{N\to\infty} \left[I + i\frac{\theta}{N}(L_3 + S_3) \right]^N \psi(x)$$

$$= \underbrace{\exp\left[i\theta\,(L_3 + S_3) \right]}\,\psi(x)$$

$= U(\theta) = e^{iJ_3\theta}$ $\Leftarrow$ generates (passive) rotation of coord.

for active rot., $\theta \to -\theta$ (i.e. rotates system).

$\uparrow$ 3rd component of $\vec{J} = \vec{L} + \vec{S}$ $\Leftarrow$ defines spin.

$\Rightarrow$ spin $= \frac{1}{2}$!

Lorentz boosts

Using ~same procedure as for rotations, find

$$S_B(\omega) = \exp\left(-\frac{\omega}{2}\gamma^0\gamma^3\right)$$

$$= I\cosh\frac{\omega}{2} - \gamma^0\gamma^3\sinh\frac{\omega}{2}$$

corresponds to Lorentz trans

$$\Lambda_B = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}$$ boost along z axis

where ω = rapidity ($\cosh\omega = \gamma$, $\sinh\omega = \beta\gamma$)

One can find $S_B^\dagger = S_B$

$$S_B^{-1} = \gamma^0 S_B^\dagger \gamma^0$$

$$\Rightarrow S_B^{-1}(\omega) = S_B(-\omega)$$ check: $\gamma^0\gamma^0 = I$, $\gamma^3\gamma^0 = -\gamma^0\gamma^3$

$\Rightarrow$ for both rotations and boosts,

$$S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0$$

(holds for any Lorentz trans. that does not include time reversal)

Illustration of Lorentz boost to find $u(p)$

Recall rest-frame spinor $u_1(0)$

$$u_1(\overset{\swarrow p^\mu=(m,0,0,0)}{0}) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

We can find $u_1(p)$ by boosting in z dir. to a frame w/ velocity $v_z = -\beta$.

$$\Rightarrow u_1(p) = S_B(\omega)\,u_1(0) = \left(I\cosh\frac{\omega}{2} + \gamma^0\gamma^3\sinh\frac{\omega}{2}\right)u_1(0)$$

where ω = rapidity of ptcl. in boosted frame

= neg. of rapidity of frame relative to ptcl. at rest.

Relate ω to E, p_z of ptcl:

$\rightarrow$ Find $\cosh\frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}$, $\sinh\frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}\cdot\frac{p_z}{E+m}$

By using $\gamma^0\gamma^3 = \begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$ and assembling ingredients,

$$\rightarrow u_1(p) = \sqrt{E+m}\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\0\end{pmatrix}$$ ~ same as what we found before for $p_x = p_y = 0$

Parity (space inversion)

$$(t', \vec{r}\,') = (t, -\vec{r}) \quad \text{i.e.}$$

$$x' = \Lambda_p x \quad \text{with} \quad \Lambda_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

same as minkowski metric

$\det \Lambda_p = -1 \longrightarrow$ "improper" Lorentz trans.

Seek $\psi'(x') = S_p \psi(x)$ such that

$$S_p^{-1} \gamma^\nu S_p = \Lambda_p{}^\nu{}_\mu \gamma^\mu$$

$$\text{rhs} = \Lambda_p{}^\nu{}_\mu \gamma^\mu = \begin{cases} \gamma^0 & \nu = 0 \\ -\gamma^i & \nu = i = 1,2,3 \end{cases}$$

Requirement satisfied with $S_p = S_p^{-1} = \gamma^0$

check: $\gamma^0 \gamma^\nu \gamma^0$

$$= \begin{cases} \gamma^0\gamma^0\gamma^0 = \gamma^0, & \nu = 0 \\ \gamma^0\gamma^i\gamma^0 = -\gamma^i, & \nu = i = 1,2,3 \end{cases}$$

Note still satisfied if mult. by arb. phase $e^{i\theta}$, or set $= 1$

Acting on Dirac spinors (check using $S_p = \gamma^0$):

$$S_p u_s(p) = + u_s(p)$$

$$S_p v_s(p) = - v_s(p)$$

$\rightarrow$ Dirac ptcl. and antiptcl. have <u>opposite parity</u>.

Bilinear Covariants

We will find amplitudes for reactions contain products of Dirac spinors, $\sim \bar{\psi}\, \Gamma\, \psi$ ← arb. 4x4 matrix

where $\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$

= "Dirac adjoint" = psi bar

Arb. 4 x 4 matrix Γ can be written as linear comb. of 16 basis matrices, choose

	matrix	N_{indep}	Lorentz trans of $\bar{\psi}\Gamma\psi$
	I_4	1	scalar
$i\gamma^0\gamma^1\gamma^2\gamma^3 \rightarrow$	γ^5	1	pseudoscalar
	γ^μ	4	vector
	$\gamma^\mu\gamma^5$	4	axial vector
$\frac{i}{2}[\gamma^\mu, \gamma^\nu] \rightarrow$	$\sigma^{\mu\nu}$	6	tensor
	total	16	

I.e. choosing this basis we will find the "bilinear covariants" $\bar{\psi}\psi$, $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, etc transform in a well-defined way under Lorentz trans.

[Later: are building blocks of Lagrangian in QFT]

Suppose ψ and $\psi^\dagger$ transform under Λ

$$\psi'(x') = \overset{S(\Lambda)}{S}\,\psi(x)$$

$$\psi'^\dagger(x') = \psi^\dagger(x)\,S^\dagger$$

Consider $\bar\psi\psi$ (scalar)

$$\bar\psi'(x')\psi'(x') = \psi'^\dagger(x')\gamma^0\psi'(x')$$

$$= \psi^\dagger(x)S^\dagger\gamma^0 S\psi(x)$$

$$= \psi^\dagger(x)\underbrace{\gamma^0\gamma^0}_{\text{insert } I}S^\dagger\gamma^0 S\,\psi(x)$$

$$= \bar\psi(x)\underbrace{\gamma^0 S^\dagger\gamma^0}_{= S^{-1}}S\,\psi(x)$$

see p 8

$$= \bar\psi(x)\psi(x)$$

$\Rightarrow$ $\bar\psi\psi$ is a Lorentz <u>scalar</u>

Next, try $\bar{\psi}\gamma^\mu\psi$:

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \psi'^\dagger(x')\gamma^0\gamma^\mu\psi'(x')$$

$$= \psi^\dagger(x)\underbrace{S^\dagger\gamma^0}_{=\gamma^0 S^{-1}}\gamma^\mu S\,\psi(x)$$

$= \gamma^0 S^{-1}$ since $S^{-1} = \gamma^0 S^\dagger \gamma^0$

$$= \psi^\dagger(x)\gamma^0 \underbrace{S^{-1}\gamma^\mu S}_{=\Lambda^\mu{}_\nu\gamma^\nu}\psi(x)$$

$= \Lambda^\mu{}_\nu\gamma^\nu$ ⇝ fundamental requirement for covar.

$$= \Lambda^\mu{}_\nu\,\bar{\psi}(x)\gamma^\nu\psi(x)$$

⇒ $\bar{\psi}\gamma^\mu\psi$ transforms like a Lorentz (four-) vector

[Example four-vec prob current $j^\mu = (\rho, \vec{j}) = \bar{\psi}\gamma^\mu\psi$]

In a similar way:

$P = \bar{\psi}\gamma^5\psi$ is a pseudo scalar

$A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ " " axial vector

$\uparrow$ $i\gamma^0\gamma^1\gamma^2\gamma^3$, $(\gamma^5)^2 = I$, $\gamma^{5\dagger} = \gamma^5$, $\{\gamma^\mu, \gamma^5\} = 0$

Under boosts & rotations, P and A^μ transform like scalar & four-vector, respectively,

But, under parity (use $\psi' = S_p\psi$, $S_p = \gamma^0$)

$$P' = \bar{\psi}'\gamma^5\psi' = \bar{\psi}\gamma^0\gamma^5\gamma^0\psi$$

$$= -\bar{\psi}\gamma^5\psi = -P$$ ⇝ pseudoscalar changes sign under parity

For <u>axial vector</u> under parity

$$A'^{\mu} = \bar{\psi}'\gamma^{\mu}\gamma^{5}\psi' = \bar{\psi}\,\overset{S_P^{\dagger}}{\gamma^{0}}\gamma^{\mu}\underbrace{\gamma^{5}\overset{S_P}{\gamma^{0}}}_{=-\gamma^0\gamma^5}\psi = -\bar{\psi}\gamma^{0}\gamma^{\mu}\gamma^{0}\gamma^{5}\psi$$

$$= \begin{cases} -\bar{\psi}\gamma^{0}\gamma^{5}\psi & \mu = 0 \\ \bar{\psi}\gamma^{i}\gamma^{5}\psi & \mu = i = 1,2,3 \quad (\text{since } \gamma^0\gamma^i = -\gamma^i\gamma^0) \end{cases}$$

$\Rightarrow$ time component picks up minus under parity

<u>Tensor</u>: $T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$

$\sigma^{\mu\nu} = -\sigma^{\nu\mu}$
$\Rightarrow$ only <u>6</u> indep. matrices

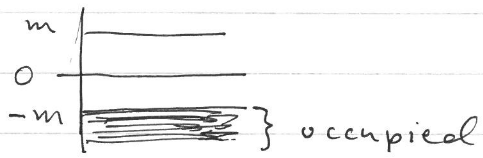
Can show under Lorentz trans Λ

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma} \quad (\underline{\text{tensor}})$$

Interpretation of negative energy sol'ns

- Dirac (1st attempt): all negative energy states <u>filled</u>

  → slide



- From Pauli principle, no transition possible to negative energy state.

- <u>If</u> neg. energy state free, e^- transition to this state possible w/ emission of γ, $E_\gamma \geq 2m_e$

- Correspondingly, γ w/ $E > 2m_e$ can lift ptcl. in neg. energy state to positive energy state, vacated "hole" = antiparticle

Cannot describe bosons, since no Pauli princ.

→ no longer use this picture in PP

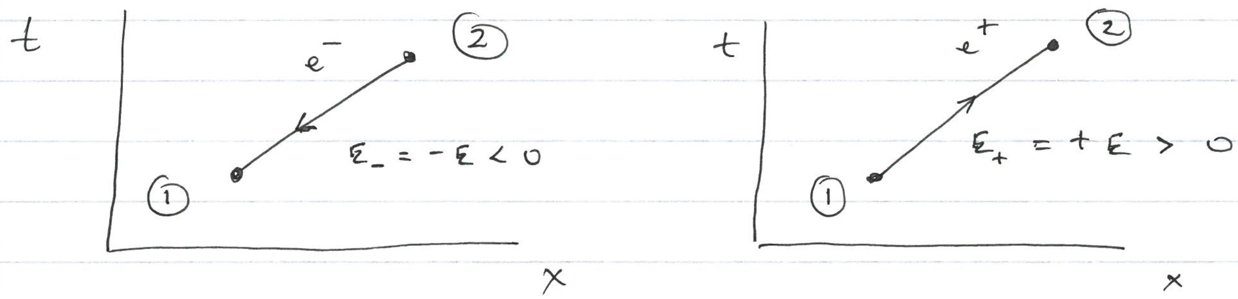
(But, e^-/hole picture used for semiconductors)

Feynman – Stückelberg interp.

Let $\Psi_- = v(p)\, e^{+ip\cdot x}$ "run backwards" in time

& interpret as antiparticle running forwards in time:

$$e^{i(-E)(-t)} = e^{iEt}$$



Also works for bosons.

Feynman interp. gives consistent description of scattering processes including ptcl. creation/annihilation

1) Emission of antiparticle w/ p^μ equivalent to absorption of particle w/ $-p^\mu$

2) Absorption of antiparticle w/ p^μ equivalent to emission of particle w/ $-p^\mu$

Dyson: mathematically equiv to QFT

Gives quick route to Feynman rules for amplitudes.

Intro to Green functions ~ see App B

Suppose we have a nonhomog DE

$$\mathcal{L}\, u(x) = f(x)$$

linear op. ↗ ↖ sol'n ↖ source

Consider first

$$\mathcal{L}\, G(x,x') = \delta(x-x')$$

↖ Green func.

Suppose (or arrange for) homogeneous BC

for u & G ($\to 0$ for $x \to \pm\infty$)

If we can find $G(x,x')$ then

claim: $u(x) = \int G(x,x')\, f(x')\, dx'$ is sol'n

proof: $\mathcal{L}\, u(x) = \mathcal{L} \int G(x,x')\, f(x')\, dx'$

$$= \int \left[\mathcal{L}\, G(x,x') \right] f(x')\, dx'$$

↖ only has derivs wrt x, not x'

$$= \int \delta(x-x')\, f(x')\, dx'$$

$$= f(x) \quad \checkmark$$

Note can always add sol'n to $\mathcal{L}\, u_0(x) = 0$

$$\to \mathcal{L}\,(u_0 + u(x)) = \underbrace{\cancel{\mathcal{L} u_0}}_{0} + \mathcal{L}\, u = f \quad \checkmark$$