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PH4442 Advanced Particle Physics Week 4

Green functions & propagators

Goal: solve $(i\not\partial - m)\psi(x) = -e \not A(x)\psi(x)$ ①

$q = -e,\ e > 0$

Dirac eq. for ptcl in EM pot. $A^\mu = (\phi, \vec{A})$

First solve

$$(i\not\partial - m) K(x, x') = \delta^4(x - x') \quad \prod_{\mu=0}^{3} \delta(x^\mu - x'^\mu)$$

4×4 matrix

depends only on $x - x'$

Once we have $K(x-x')$ ~ Green func. of $(i\not\partial - m)$

claim: $\psi(x) = \phi(x) - e\int K(x-x')\not A(x')\psi(x')\,d^4x'$ ②

sol'n to $(i\not\partial - m)\phi(x) = 0$

check: $(i\not\partial - m)\psi(x) = \cancel{(i\not\partial - m)\phi(x)}$

∂ only acts on x, not x'

$$- e\int \underbrace{(i\not\partial - m) K(x-x')}_{\delta^4(x-x')} \not A(x')\psi(x')\,d^4x'$$

assume bounds of $\int$ indep. of x

$= -e\not A(x)\psi(x)$ ✓

Note r.h.s. contains sol'n $\psi(x)$

⇒ ② not formula for sol'n to ①, but an integral eq. that the solution satisfies

Solve iteratively → expansion in powers of e

→ Feynman diagrams, representing terms in perturbation series

Electron propagator

We have

$$\psi(x) = \underset{\substack{\uparrow \\ (i\not\partial - m)\phi = 0}}{\phi(x)} - e\int \underset{\substack{\uparrow \\ (i\not\partial - m)K(x-x') = \delta^4(x-x')}}{K(x-x')} \not A(x-x')\,\psi(x')\,d^4x'$$

Solve by regarding $-e\int K(x-x')\not A(x')\psi(x')\,d^4x'$

as a perturbation, i.e.,

$$\psi^{(0)}(x) = \phi(x)$$

$$\psi^{(1)}(x) = \phi(x) - e\int d^4x'\,K(x-x')\not A(x')\underbrace{\phi(x')}_{\psi^{(0)}(x')}$$

$$\psi^{(2)}(x) = \phi(x)$$
$$- e\int d^4x'\,K(x-x')\not A(x')\underbrace{\left[\phi(x') - e\int d^4x''\,K(x'-x'')\not A(x'')\phi(x'')\right]}_{\psi^{(1)}(x')}$$

= no scatter + single scatter

+ double scatter + ---

In general

$$\psi^{(n+1)}(x) = \phi(x) - e\int d^4x'\,K(x-x')\not A(x')\,\psi^{(n)}(x')$$

Computing the Green function

Fourier transform of $K(x-x')$

$$\tilde{K}(p) = \int d^4x \; e^{ip\cdot(x-x')} K(x-x')$$

$$K(x-x') = \frac{1}{(2\pi)^4}\int d^4p \; e^{-ip\cdot(x-x')} \tilde{K}(p)$$

$$\Rightarrow (i\not{\partial}-m)K(x-x') = \frac{1}{(2\pi)^4}\int d^4p \; e^{-ip\cdot(x-x')} (\not{p}-m)\tilde{K}(p)$$

$$= \delta^4(x-x') \quad \text{by def.}$$

$$= \frac{1}{(2\pi)^4}\int d^4p \; e^{-ip\cdot(x-x')}$$

integral rep. of δ function

$$\Rightarrow (\not{p}-m)\tilde{K}(p) = I$$

4x4 unit matrix

mult. on left by $\not{p}+m$

$$(\not{p}+m)(\not{p}-m)\tilde{K}(p) = (p^2-m^2)\tilde{K}(p) = \not{p}+m$$

$$\Rightarrow \boxed{\tilde{K}(p) = \frac{\not{p}+m}{p^2-m^2}} \qquad (p^2 \neq m^2)$$

= electron propagator

Note defined only for virtual electron $(p^2 \neq m^2)$

Invert Fourier trans. to obtain Green func,

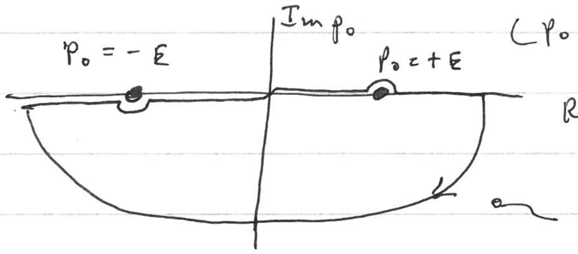
$$K(x-x') = \frac{1}{(2\pi)^4}\int d^4p \; e^{-ip\cdot(x-x')}\,\tilde{K}(p)$$

$$= \frac{1}{(2\pi)^4}\int d^3p \; e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\int_{-\infty}^{\infty} \frac{e^{-ip_0(t-t')}(\not{p}+m)}{(p_0-E)(p_0+E)}$$

$E \equiv +\sqrt{|\vec{p}|^2+m^2}\,, \quad -\infty < p_0 < \infty$

i.e. $p^2 - m^2 = p_0^2 - |\vec{p}|^2 - m^2 = p_0^2 - E^2 = (p_0-E)(p_0+E)$

Integrand has poles at $p_0 = \pm E$



Justify choice of contour below

note clockwise

In lower half-plane, p_0 is negative imaginary,

i.e. $e^{-ip_0(t-t')} \longrightarrow 0$ if $t > t'$

Using residue theorem, pole at $p_0 = +E$

$$\oint \underbrace{f(z)}_{\frac{g(z)}{h(z)}}\, dz = \int dp_0 \underbrace{\frac{1}{p_0 - E}}_{h(p_0)} \underbrace{\left[\frac{(\not{p}+m)\,e^{-ip_0(t-t')}}{p_0+E}\right]}_{g(p_0)} = -2\pi i\, g(p_0 = E)$$

because contour clockwise

residue at $p_0 = E$

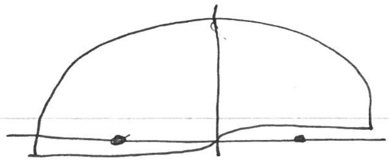
$$\Rightarrow K(x-x') = \frac{-i}{(2\pi)^3}\int d^3p \, \exp\left[i\vec{p}\cdot(\vec{x}-\vec{x}') - iE(t-t')\right] \times \frac{\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m}{2E}$$

$+\sqrt{|\vec{p}|^2+m^2}$

1

$(t > t')$

For $t < t'$ use contour



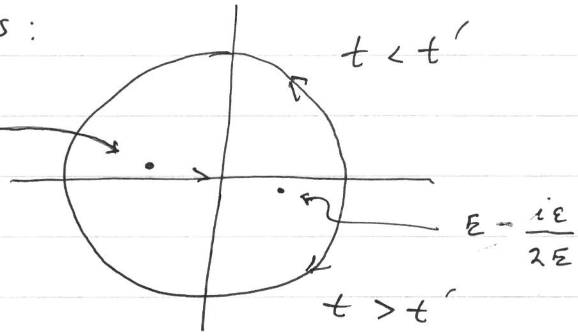
which gives

!

(2) $$K(x-x') = -\frac{i}{(2\pi)^3}\int d^3p \exp\left[i\vec{p}\cdot(\vec{x}-\vec{x}') - iE(t-t')\right] \cdot \frac{-\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m}{2E}$$

i.e. paths of integration chosen so that one gives $K(x-x')$ for $t > t'$, the other for $t < t'$.

Equivalently, move the poles:



$$\Rightarrow \tilde{K}(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

$\varepsilon > 0$

Time-ordered or "Feynman" propagator

($\rightarrow$ slides) Propagator and time evolution

Consider free ptcl $\vec{k} = (k_0, \vec{k})$ with $k_0 > 0$

$$\phi_+(x') = u(k)\exp\left[-ik_0 t' + i\vec{k}\cdot\vec{x}\right] \qquad (3)$$

pos. energy

Claim: at later time $t > t'$, wave function is

$$\phi_+(x) = \phi_+(t,\vec{x}) = i\int d^3x' \underbrace{K(x-x')}_{\text{use (1)}} \gamma^0 \phi_+(t',\vec{x}')$$

($\rightarrow$ slides)

Proof: use ① for $K(x-x')$ ← for $t>t'$ and ③ in ④

$$\Rightarrow RHS = i\int d^3x' \left[\frac{-i}{(2\pi)^3} \int d^3p \, \exp\left[i\vec{p}\cdot(\vec{x}-\vec{x}') - iE(t-t')\right] \frac{\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m}{2E} \right.$$

$$\left. \times \, \gamma^0 u(k) \exp(-ik_0 t' + i\vec{k}\cdot\vec{x}') \right]$$

Collect x' terms, move out x "

$$= \int d^3p \, \underbrace{\frac{1}{(2\pi)^3} \int d^3x' \exp\left(i(\vec{k}-\vec{p})\cdot\vec{x}'\right)}_{\delta^3(\vec{k}-\vec{p})} \frac{\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m}{2E} \exp\left(i(E-k_0)t'\right)$$

$$\times \, \gamma^0 u(k) \exp(i\vec{p}\cdot\vec{x} - Et)$$

$E = \sqrt{|\vec{p}|^2 + m^2}$ gets integrated away

$$= \exp\left(-ik_0 t + i\vec{k}\cdot\vec{x}\right) \times \frac{\left[(\gamma^0 k_0 - \vec{\gamma}\cdot\vec{k} + m)\gamma^0 u(k) \right]}{2k_0}$$

since $\vec{k} = \vec{p}$, $k_0 = E$

$u(k)$ is sol'n of Dirac eq.

$$(\not{k} - m)u(k) = 0 = (\gamma^0 k_0 - \vec{\gamma}\cdot\vec{k} - m)u(k)$$

$$\Rightarrow [\;] = (\gamma^0 k_0 - \vec{\gamma}\cdot\vec{k} + m)\gamma^0 u(k)$$

$$= \gamma^0(\gamma^0 k_0 + \vec{\gamma}\cdot\vec{k} + m)u(k) \quad \text{since } \gamma^i\gamma^0 = -\gamma^0\gamma^i$$

$$= \underbrace{(\gamma^0)^2}_{=I} k_0 u(k) + \gamma^0 \underbrace{(\vec{\gamma}\cdot\vec{k} + m)u(k)}_{=\gamma^0 k_0 u(k) \text{ from}}$$

$$= 2k_0 u(k)$$

$$\Rightarrow RHS = u(k)\exp(-ik_0 t + \vec{k}\cdot\vec{x})$$

$$= \phi_+(t, \vec{x}) \quad \checkmark$$

(→ slides)

For times $t < t'$, need ② for $K(x-x')$

$$K(x-x') = \frac{-i}{(2\pi)^3} \int d^3p \, \exp\left[i\vec{p}\cdot(\vec{x}-\vec{x}') + iE(t-t') \right] \times \frac{-\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m}{2E}$$

This gives term $(-\gamma^0 k_0 - \vec{\gamma}\cdot\vec{k} + m)\, u(k) = -(\not{k} - m)\, u(k) = 0$

i.e. for $k_0 > 0$,

$$i \int d^3x' \, K(x-x')\, \gamma^0 \phi_+(t', \vec{x}') = \begin{cases} \phi_+(t, \vec{x}) & t > t' \\ 0 & t < t' \end{cases}$$

i.e. for $k_0 > 0$, $\phi_+(t,\vec{x})$ is given as an integral over d^3x' at a given time t' in the past $(t > t')$

Similarly, for neg. energy sol'n $E = -k_0 < 0$ use

$\phi_-(x') = v(k)\, e^{ik\cdot x'}$, find same as above but

$t > t' \leftrightarrow t < t'$

i.e. positive energy sol'ns spread out into future,

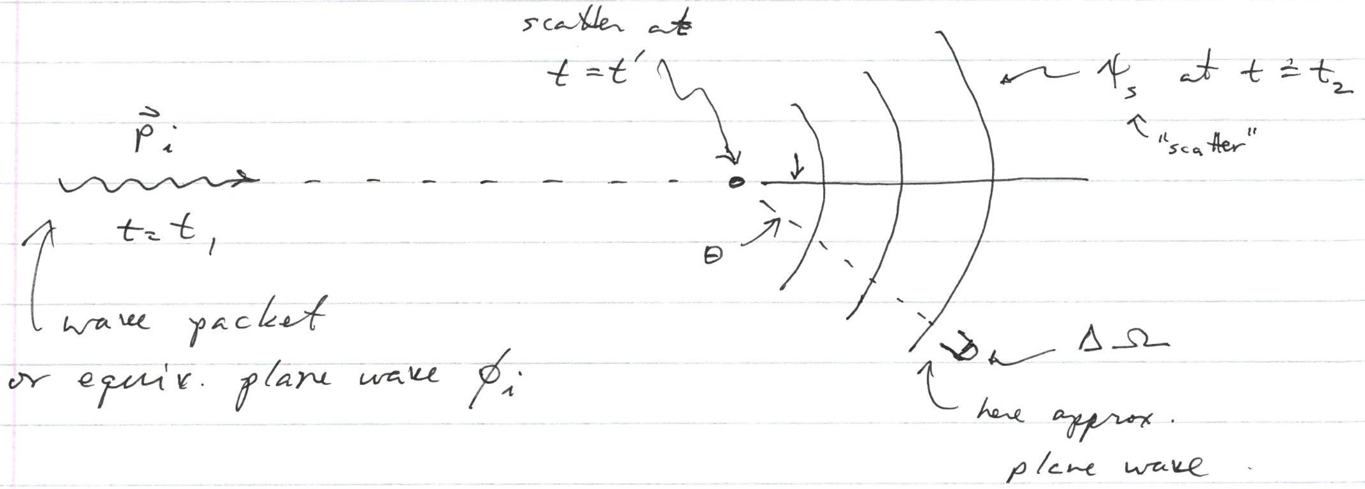
negative " " " " " past

In a similar way find useful formula for adjoint $\bar{\phi}$ (for positive energy)

$$\left.\begin{matrix} \overset{t' < t}{\bar{\phi}_+(t', \vec{x}')} \\ \underset{t' > t}{0} \end{matrix}\right\} = i \int d^3x \, \bar{\phi}_+(t, \vec{x})\, \gamma^0 K(x-x')$$

integral over d^3x at later time $t > t'$

Matrix element for e^- scattering
by EM potential A^μ (either fixed or from other ptcl.)



$t = t_1$, no interaction w/ A^μ

$t = t'$, interaction

$t = t_2$, no interaction, outgoing ptcl. detected

Define <u>S matrix</u> by $\psi_s = S \phi_i$

S operator

To 2^{nd} order (see p. 2)

$$\left.\psi_s^{(2)}(x) = \phi_i(x) - e\int d^4x'\, K(x-x')\, \not{A}(x')\, \phi_i(x')\right\}\ \psi_s^{(1)}(x)$$

$$+\ e^2 \iint d^4x'\, d^4x''\, K(x-x')\, \not{A}(x'')\, K(x''-x')\, \not{A}(x')\, \phi_i(x')$$

S_{fi} (defined by $\psi_s = S\phi_i$) is

$$\underbrace{S_{fi}}_{\text{S matrix}} = \langle \vec{p}_f, t_f | \psi_s \rangle = \langle \vec{p}_f, \overset{+\infty}{t_f} | S | \vec{p}_i, \overset{-\infty}{t_i} \rangle$$

$$= \int d^3x\, \phi_f^\dagger(t_f, \vec{x})\, S\, \phi_i(t_i, \vec{x})$$ or i, f refer to $\vec{p}_i, \vec{p}_f$

$$= \int d^3x\, \phi_f^\dagger(x)\, \phi_i(x) - e\int d^3x\, \phi_f^\dagger(x) \int d^4x'\, K(x-x') \not{A}(x') \phi_i(x')$$

means $(t_f, \vec{x})$

$$+ e^2 \int d^3x\, \phi_f^\dagger(x) \int d^4x'\, d^4x''\, K(x-x'') \not{A}(x'')\, K(x''-x') \not{A}(x')\, \phi_i(x')$$

$(t_f, \vec{x})$

Use $\phi_i(x) = u(p_i)\, e^{-ip_i\cdot x} = u(p_i)\, e^{-i(p_i^0 t_f - \vec{p}_i\cdot\vec{x})}$

$\phi_f(x) = u(p_f)\, e^{-ip_f\cdot x} = u(p_f)\, e^{-i(p_f^0 t_f - \vec{p}_f\cdot\vec{x})}$

$\Rightarrow \phi_f^\dagger(x) = \bar{u}(p_f)\gamma^0 e^{ip_f\cdot x} = \bar{u}(p_f)\gamma^0 e^{i(p_f^0 t_f - \vec{p}_f\cdot\vec{x})}$

$$\Rightarrow S_{fi}^{(0)} = \int d^3x\, \phi_f^\dagger(x)\, \phi_i(x)$$

$$= \underbrace{\int d^3x\, e^{i(\vec{p}_i - \vec{p}_f)\cdot\vec{x}}}_{(2\pi)^3 \delta^3(\vec{p}_i - \vec{p}_f)}\, e^{-i(p_{0i} - p_{0f})t}\, \underbrace{\bar{u}(p_f)\gamma^0 u(p_i)}_{2E_i}$$

$(2\pi)^3 \delta^3(\vec{p}_i - \vec{p}_f)$ → 0 unless $\vec{p}_i = \vec{p}_f$

$\Rightarrow$ set $p_{0i} = p_{0f} \equiv E_i$

$\int e^0 d^3x \to V$, cancels $\frac{1}{\sqrt{V}}$ factors of $\bar{u}, u$

$\Rightarrow \bar{u}(p_i)\gamma^0 u(p_i) = u^\dagger(p_i)\, u(p_i) = 2E_i$

$\Rightarrow S_{fi} = 2E_i\delta_{fi} + S_{fi}^{(1)} + S_{fi}^{(2)} + \cdots$

"no scattering", will not enter into transition prob, cross section, etc

1st order matrix element

$$S^{(1)}_{fi} = -e\int d^4x' \int d^3x \; \phi^\dagger_f(x)\; K(x-x')\; \cancel{A}(x')\; \phi_i(x')$$

$x = (t_f, \vec{x})$, $t_f \to \infty$

Use $\bar{\phi}(t', \vec{x}') = i\int d^3x \; \bar{\phi}(t,\vec{x})\, \gamma^0 K(x-x')$

see 5 p7,
swap $x' \leftrightarrow x$

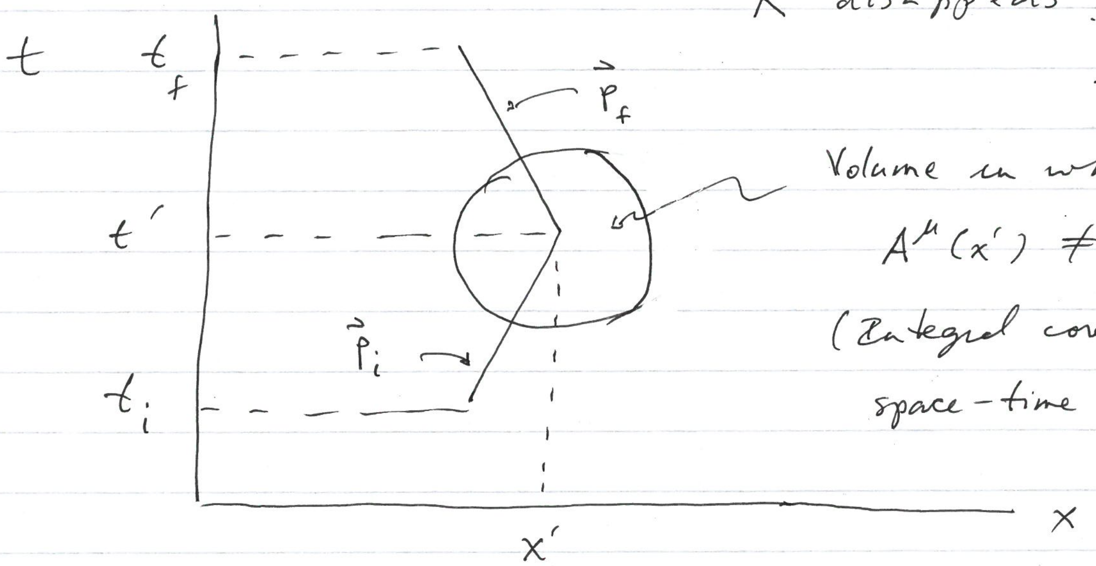
$$= i\int d^3x \; \phi^\dagger(t,\vec{x})\, K(x-x') \quad \text{for} \quad t' < t$$

$$\Rightarrow \int d^3x \; \phi^\dagger_f(x)\, K(x-x') = -i\,\bar{\phi}_f(x') \qquad t' < t$$

time of interaction (t') — time of detection (t)

$$\Rightarrow \boxed{S^{(1)}_{fi} = ie\int d^4x' \; \bar{\phi}_f(x')\, \cancel{A}(x')\, \phi_i(x')}$$

K disappears!



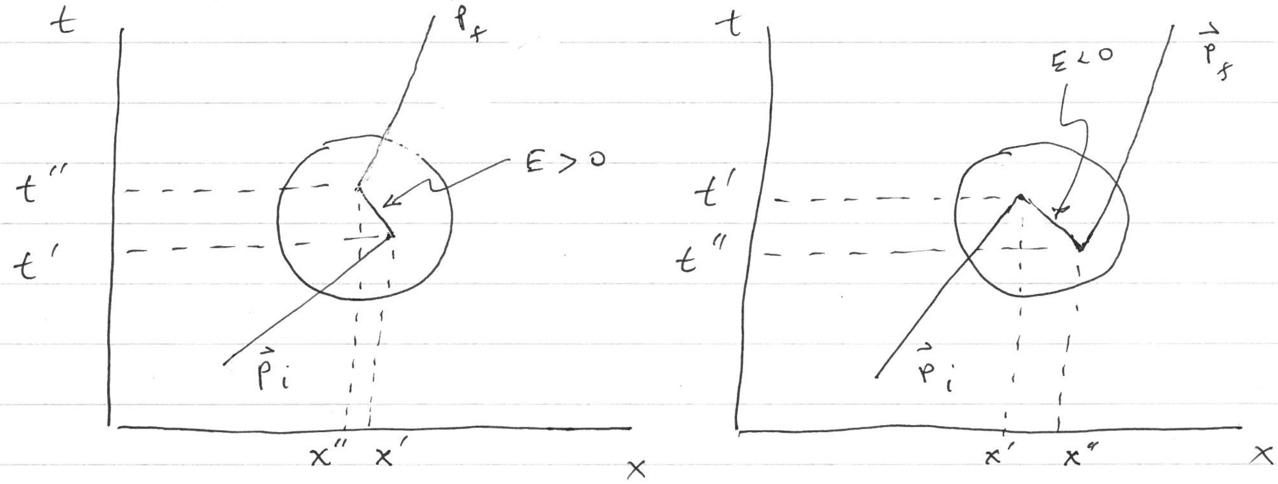
Volume in which $A^\mu(x') \neq 0$ (Integral covers this space-time vol.)

(→ slides)

2nd order matrix element

$$S^{(2)}_{fi} = e^2 \iint d^4x'\, d^4x'' \underbrace{\int d^3x\, \phi^\dagger_f(x)\, K(x-x'')}_{\leftarrow\, -i\bar{\phi}_f(x'')} \not{A}(x'')\, K(x''-x')\, \not{A}(x')\, \phi_i(x')$$

$$= -ie^2 \iint d^4x'\, d^4x''\, \bar{\phi}_f(x'')\, \not{A}(x'')\, K(x''-x')\, \not{A}(x')\, \phi_i(x')$$



double scatter

pair creation + annihilation

Both diagrams are included in $S^{(2)}_{fi}$ since double integral contains both $t''>t'$ and $t''<t'$

For $t''>t'$, $K(x''-x')$ gives propagation of e^- with $E>0$ ~~from~~ x' to x''

For $t''<t'$, $K(x''-x')$ represents propagation of e^- with $E<0$ from x' to x'', or equivalently of e^+ with $E>0$ from x'' to x'

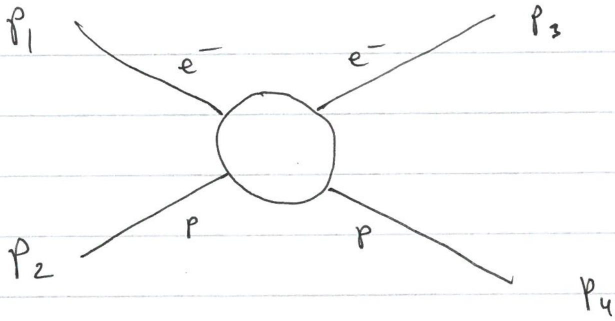
If one diagram left out, result is not Lorentz invariant.

Photon propagator

For scattering of 2 charged particles, compute A^μ from one of them & use in formula for S_{fi} for scattering of the other (same ans. if roles reversed)

E.g. $e^- p \to e^- p$

↗ "Dirac proton" (no internal structure)



Compute A^μ created by proton

$$\partial_\alpha \partial^\alpha A^\mu(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu(x) = \underbrace{e\, J^\mu(x)}_{J^\mu_{e.m.}}$$

Maxwell eq. w/ Lorentz gauge condition $\partial_\mu A^\mu = 0$

Green func. for $\partial_\alpha \partial^\alpha$ defined by

$$\partial_\alpha \partial^\alpha D^{\mu\nu}(x-x') = g^{\mu\nu} \delta^4(x-x')$$

Once $D^{\mu\nu}$ found, A^μ obtained from

$$A^\mu(x) = e \int d^4x' \, D^{\mu\nu}(x-x') \, J_\nu(x')$$

looks like A^μ will depend on all components of J_ν

Check: $\partial_\alpha \partial^\alpha A^\mu(x) = e \int d^4x' \, \partial_\alpha \partial^\alpha D^{\mu\nu}(x-x') \, J_\nu(x')$

$$= e \int d^4x' \, \delta^4(x-x') \, g^{\mu\nu} J_\nu(x')$$

$$= e\, J^\mu(x) \quad \checkmark$$

$D^{\mu\nu}(x)$ related to Fourier trans $\tilde{D}^{\mu\nu}(q)$ by

$$D^{\mu\nu}(x-x') = \int \frac{d^4q}{(2\pi)^4}\, \tilde{D}^{\mu\nu}(q)\, e^{-iq\cdot(x-x')}$$

$$\Rightarrow \partial_\alpha \partial^\alpha D^{\mu\nu}(x-x') = \int \frac{d^4q}{(2\pi)^4}\, \tilde{D}^{\mu\nu}(q)\,(-q^2)\, e^{-iq\cdot(x-x')}$$

$$= g^{\mu\nu}\,\delta^4(x-x') \quad \text{by definition}$$

$$= g^{\mu\nu} \underbrace{\int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot(x-x')}}_{\text{integral representation of delta func.}}$$

compare to

$$\Rightarrow \tilde{D}^{\mu\nu}(q) = -\frac{g^{\mu\nu}}{q^2}$$

$$\longrightarrow -\frac{g^{\mu\nu}}{q^2+i\varepsilon}, \quad \varepsilon > 0$$

(argument ~ same as for e^-)

So in fact, $D^{\mu\nu} \propto g^{\mu\nu}$

$\Rightarrow A^\mu$ depends only on μ component of J^μ

back to ep scattering example

Recall prob. current for a Dirac ptcl. is

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

same four-momenta

But now for proton have

$$\psi_2(x) = u(p_2)\,e^{-ip_2\cdot x} \qquad \text{initial state}$$

$$\psi_4(x) = u(p_4)\,e^{-ip_4\cdot x} \qquad \text{final ''}$$

Guess: $J^\mu(x) = \bar{\psi}_4(x)\gamma^\mu\psi_2(x)$ = "transition current"

Can justify by requirement that we get same answer if p is projectile & e^- is target.

$$\Rightarrow \quad e\,J^\mu(x) = e\,\bar{u}(p_4)\gamma^\mu u(p_2)\,e^{i(p_4-p_2)\cdot x}$$

$$\Rightarrow \quad A^\mu(x) = e\int d^4x'\,D^{\mu\nu}(x-x')\,J_\nu(x')$$

$$= e\int d^4x'\int\frac{d^4q}{(2\pi)^4}\,\frac{-g^{\mu\nu}}{q^2+i\varepsilon}\,e^{-i(p_4-p_2+q)\cdot x'}\,e^{-iq\cdot x}\,\bar{u}(p_4)\gamma_\nu u(p_2)$$

$$= e\int d^4q\,\delta^4(p_4-p_2+q)\,\frac{-g^{\mu\nu}}{q^2+i\varepsilon}\,e^{-iq\cdot x}\,\bar{u}(p_4)\gamma_\nu u(p_2)$$

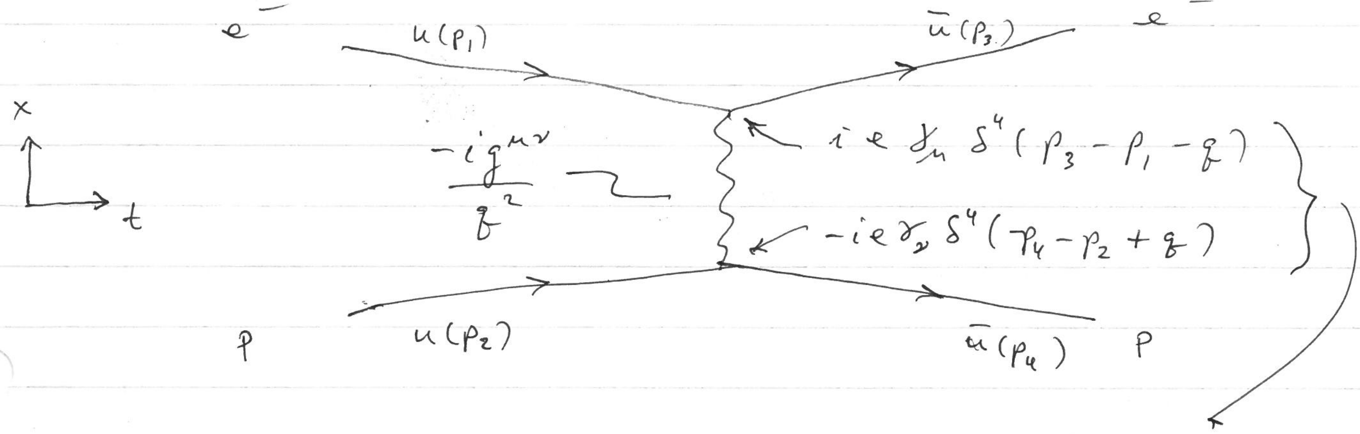
use in formula for S_{fi}

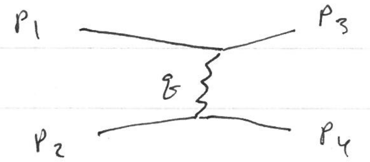
$$S_{fi}^{(1)} = ie\int d^4x\,\bar{\phi}_f(x)\,\not{A}(x)\,\phi_i(x) \quad \text{w/} \quad \phi_i(x) = u(p_1)\,e^{-ip_1\cdot x}$$

$$\phi_f(x) = u(p_3)\,e^{-ip_3\cdot x}$$

$$\Rightarrow S_{fi}^{(1)} = ie^2 \int d^4q \; \delta^4(p_4 - p_2 + q)\, \delta^4(p_3 - p_1 - q)\, (2\pi)^4$$

$$\times \; \bar{u}(p_3)\gamma_\mu u(p_1) \frac{-g^{\mu\nu}}{q^2 + i\varepsilon} \bar{u}(p_4)\gamma_\nu u(p_2)$$

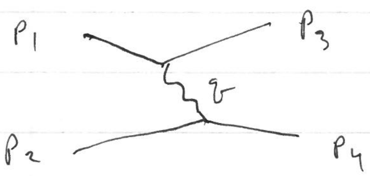




$p_1 + q = p_3$

$p_2 = q + p_4$,

4-momentum conserved at each vertex.



$p_1 = q + p_3$

$q + p_2 = p_4$

Equivalently, $q \to -q$ in $\int$

both time orderings implicitly included (define q using either)

From $\int d^4q \; \delta^4(p_4 - p_2 + q)\, \delta^4(p_3 - p_1 - q) = \delta^4(p_3 + p_4 - p_1 - p_2)$

$\longleftrightarrow$ overall 4-momentum conservation

$\longrightarrow$ to do $\int d^4q$, just impose conservation of 4-momentum at each vertex (& hence total)

$$S_{fi}^{(1)} = (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)\left[\bar{u}(p_3)(ie\gamma_\mu)u(p_1)\right]\frac{-i}{q^2}\left[\bar{u}(p_4)(-ie\gamma^\mu)u(p_2)\right]$$

Define (Lorentz) invariant amplitude $\mathcal{M}$ through

$$S_{fi} = (2\pi)^4 \delta^4(P_f - P_i)\,\mathcal{M}$$

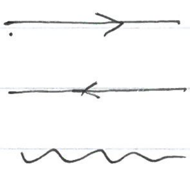
phase convention: sometimes $\mathcal{M} \to i\mathcal{M}$ or $-i\mathcal{M}$

$q^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2$

$$\text{i.e. } \mathcal{M} = \underbrace{-e\,\bar{u}(p_3)\gamma_\mu u(p_1)}_{-eJ_{\mu,\,\mathrm{elec}}} \times \underbrace{\frac{-i}{q^2}}_{\text{photon propagator}} \times \underbrace{e\,\bar{u}(p_4)\gamma^\mu u(p_2)}_{eJ^\mu_{\mathrm{proton}}}$$

start

Rules for Feynman diagrams (QED)


fermion
antifermion
photon

Incoming (outgoing) fermion $u(p)$ $(\bar{u}(p))$

" " antifermion $v(p)$ $(\bar{v}(p))$

" " photon $\varepsilon_\mu(k)$ $(\varepsilon^*_\mu(k))$

polarisation vector

virtual photon $\dfrac{-ig^{\mu\nu}}{q^2+i\varepsilon}$

intermediate

virtual electron $\dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$

For this course, won't need ε

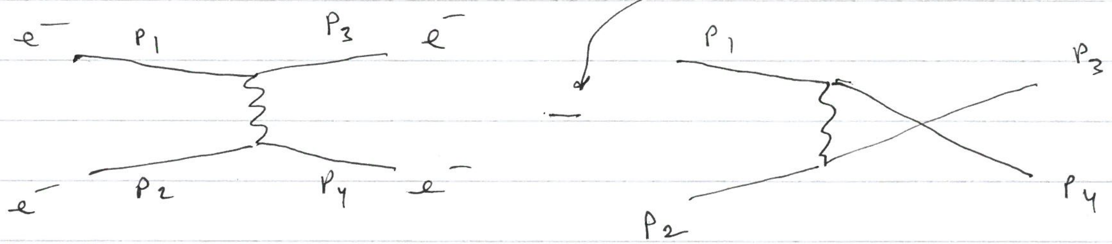
fermion $-\gamma$ vertex: $-i\underbrace{(\pm e)}_{\xi}\gamma_\mu$

Impose 4-momentum cons. at every vertex.

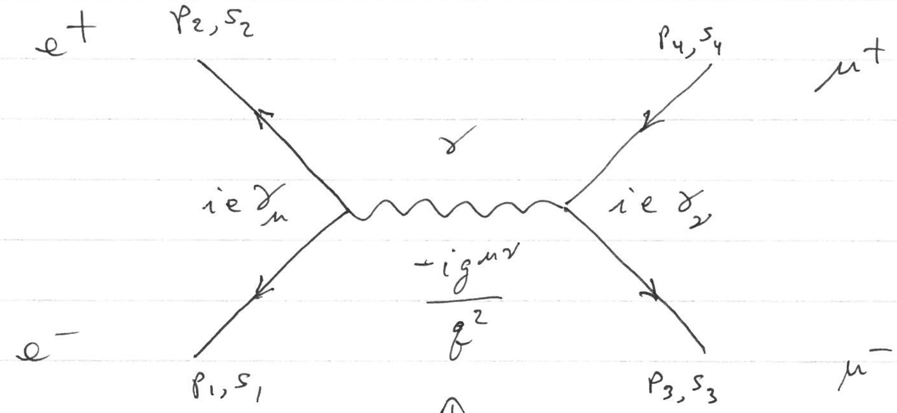
⚠ Need to keep track of phase factors when adding diagrams.

For a diagram that differs from another by exchange of an identical fermion, include a minus sign

e.g.



Example: $e^+e^- \to \mu^+\mu^-$ in QED



$q = p_1 + p_2 = p_3 + p_4$ } 4-momentum conserved at each vertex

Follow the fermion lines backwards,

u for particle, v for antiparticle

"bar" at start of fermion line, no bar at end

$$\mathcal{M} = \underset{e^+}{\bar{v}(p_2, s_2)} (ie\gamma_\mu) \underset{e^-}{u(p_1, s_1)} \underset{\gamma}{\frac{-ig^{\mu\nu}}{q^2}} \underset{\mu^-}{\bar{u}(p_3, s_3)} (ie\gamma_\nu) \underset{\mu^+}{v(p_4, s_4)}$$

contract $g^{\mu\nu}\gamma_\nu = \gamma^\mu$

$$\mathcal{M} = i\frac{e^2}{q^2} \underbrace{\left[\bar{v}(p_2, s_2)\gamma_\mu u(p_1, s_1)\right]}_{\substack{j^\mu \\ \text{electron current}}} \underbrace{\left[\bar{u}(p_3, s_3)\gamma^\mu v(p_4, s_4)\right]}_{\substack{J^\mu \\ \text{proton current}}}$$

$q^2 = (p_1 + p_2)^2$

$= E_{cm}^2$

$(= s)$

(Appendix) Integral representation of δ function

Take Fourier transform of $\delta(x-x')$

$$\hat{\delta}(k) = \int \delta(x-x')\, e^{-ikx}\, dx$$

$$= e^{-ikx'}$$

Take inverse Fourier transform

$$\delta(x-x') = \frac{1}{2\pi}\int e^{ik(x-x')}\, dk$$

Note $\delta(x-x') = \delta(x'-x) \longrightarrow$ either sign ok in exponent

Often use 4-D version

$$\delta^4(x-x') = \frac{1}{(2\pi)^4}\int e^{i\underbrace{k\cdot(x-x')}_{\text{4-vectors}}}\, d^4k$$