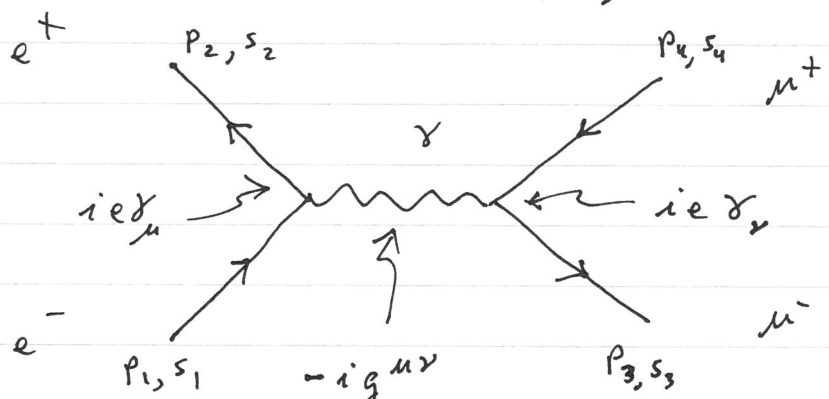
G. Cowan 31.10.25

PH4442 Advanced Particle Physics Week 5

Summarize rules for Feynman diagrams → slides

Example: $e^+ e^- \to \mu^+ \mu^-$ in QED



$$\frac{-ig^{\mu\nu}}{q^2} \qquad q = p_1 + p_2 = p_3 + p_4$$

4-mom. conserved at each vertex

- Follow the fermion lines backwards;
- u for fermion, v for antifermion,
- "bar" at start of fermion line, no bar at end
- vertex factor w/ coupling at each vertex
- propagator for all internal lines

contract Lorentz indices as appropriate

$$\mathcal{M} = \underset{e^+}{\bar{v}(p_2,s_2)}(ie\gamma_\mu)\underset{e^-}{u(p_1,s_1)} \frac{-ig^{\mu\nu}}{q^2} \underset{\mu^-}{\bar{u}(p_3,s_3)}(ie\gamma_\nu)\underset{\mu^+}{v(p_4,s_4)}$$

contract $g^{\mu\nu}\gamma_\nu = \gamma^\mu$

$$\mathcal{M} = \frac{ie^2}{q^2} \underbrace{\left[\bar{v}(p_2,s_2)\gamma_\mu u(p_1,s_1)\right]}_{J^\mu_{elec}} \underbrace{\left[\bar{u}(p_3,s_3)\gamma^\mu v(p_4,s_4)\right]}_{J^\mu_\mu}$$

$q^2 = (p_1+p_2)^2$

$= E_{cm}^2 \equiv s$

From amplitudes to cross sections & decay rates

(Notes Ch. 7)

From an amplitude $\mathcal{M}$, we want:

$$d\Gamma = \frac{|\mathcal{M}|^2}{2M} \, dLIPS$$

decay rate
$[\Gamma]$ = energy
= 1/time

$$d\sigma = \frac{|\mathcal{M}|^2}{F} \, dLIPS$$

cross section
$[\sigma]$ = energy^{-2} = area

flux factor

Lorentz invariant phase space

Recall from last week:

$$T_{fi} = \langle f, t_f \to \infty \,|\, U(t_f, t_i) \,|\, i, t_i \to -\infty \rangle$$

transition amplitude, like S_{fi} but drop the no-interaction term

$$= (2\pi)^4 \, \delta^4 (P_f - P_i) \, \mathcal{M}$$

total i/f state

invariant amplitude

Cannot yet treat $|T_{fi}|^2$ as a probability because wave functions in $\mathcal{M}$ not normalized to unity,

e.g., $\psi(x) = u(p) \, e^{-ip\cdot x}$

$$\to \int_V \psi^\dagger(x) \, \psi(x) \, d^3x = \underbrace{u^\dagger(p) \, u(p)}_{2E} \int_V e^{\overbrace{i(p-p)\cdot x}^{=0}} \, d^3x$$

normalizing volume

$$= \underline{2EV}$$

$\Rightarrow$ Need to divide $\mathcal{M}$ by

$$\sqrt{\langle j | j \rangle} = \sqrt{2E_j V} \quad \text{for all wave functions}$$

same for bosons, also normalized to $2EV$

in initial & final states

$$N_i + N_f = N$$

ptcls in initial, # ptcls in final states

normalized

$$\Rightarrow \quad T_{fi} = (2\pi)^4 \delta^4(P_f - P_i) \frac{\mathcal{M}}{\prod_{i=1}^{N} \sqrt{2E_i V}}$$

$$\text{Probability}(f \leftarrow i) = P_{fi} = |T_{fi}|^2$$

$$= \frac{|\mathcal{M}|^2}{\prod_{i=1}^{N} 2E_i V} \left| (2\pi)^4 \delta^4(P_f - P_i) \right|^2$$

For square of delta function, represent one of them as

$$(2\pi)^4 \delta^4(P_f - P_i) = \int_{V,T} e^{i \underbrace{(P_f - P_i)}\cdot x} d^4x$$

$t_f - t_i$

other $\delta^4(P_f - P_i)$ enforces $P_f = P_i$

$$\rightarrow \int_{V,T} e^0 d^4x = VT$$

$$\Rightarrow \quad \frac{P_{fi}}{T} = \frac{|\mathcal{M}|^2 (2\pi)^4 \delta^4(P_f - P_i)}{V^{N-1} \prod_{i=1}^{N} 2E_i}$$

P_f, P_i exactly given

$$\frac{P_{fi}}{T} = \frac{\text{prob}}{\text{time}} \quad \text{for } \underline{\text{specific } f} \leftarrow \underline{\text{specific } i}$$

We can set a specific initial state, but final state ptcls take on values p_f from a continuum of possible momenta.

$\Rightarrow$ only meaningful to predict prob. density

$$\sim P(p_f \in [p_f, p_f + dp_f]) = f(p_f)\, dp_f$$

$\delta^4(P_f - P_i)$ restricts allowed momenta to subspace consistent w/ total energy/momentum conservation.

System is in (large) box of size $V = L^3$

Periodic B.C. $\rightarrow$ momenta restricted to $p = \frac{2\pi n}{L}$ (integer n) $\quad (\hbar = 1)$

$\Rightarrow$ momentum spacing for each component $\Delta p = \frac{2\pi}{L}$

$\Rightarrow$ one allowed state in phase-space vol. (6-D $(\vec{p}, \vec{x})$)

$$\text{of } \underbrace{\left(\frac{2\pi}{L}\right)^3}_{\Delta p_x \cdot \Delta p_y \cdot \Delta p_z} \cdot V = \underline{(2\pi)^3}$$

$\Rightarrow$ in phase-space element $V\, d^3\vec{p}_j$ for ptcl. j

there are $dn_j = \dfrac{V\, d^3\vec{p}_j}{(2\pi)^3}$ states

$\Rightarrow$ Include factor dn_j for each of the

N_f final-state ptcls $\rightarrow$ Fermi's Golden Rule:

$$\boxed{\frac{dP}{T} \equiv \underbrace{dW}_{\propto \text{ pdf } f(\underbrace{\vec{p}_1, \dots, \vec{p}_{N_f}}_{\text{final}} \,|\, \underbrace{\vec{p}_1, \dots, \vec{p}_{N_i}}_{\text{initial}})} = \frac{|\mathcal{M}|^2 (2\pi)^4 \delta^4(P_f - P_i)}{V^{N_i - 1} \prod_{j=1}^{N_i} 2E_j} \prod_{k=1}^{N_f} \frac{d^3\vec{p}_k}{(2\pi)^3\, 2E_k}}$$

The factors

$$dLIPS = (2\pi)^4 \delta^4(P_f - P_i) \prod_{k=1}^{N_f} \frac{d^3\vec{p}_k}{(2\pi)^3\, 2E_k} \qquad \left(= d\Phi_{N_f}\right)$$

are called the Lorentz Invariant $\underbrace{\text{Phase Space}}_{(3N_f - 4)\text{-D subspace of } 3N_f\text{-D } \vec{p} \text{ space.}}$ (factor)

Claim: dLIPS invariant under Lorentz trans.

We will need to find

$$\text{Rate} \sim \int dW = \int \frac{|\mathcal{M}|^2\, dLIPS}{V^{N_i - 1} \prod_{j=1}^{N_i} 2E_j}$$

In general, $|\mathcal{M}|^2$ depends on all $\vec{p}_k$,

$\Rightarrow$ cannot pull out of integral

$\Rightarrow$ complicated numerical integration for $N_f \geq 3$.

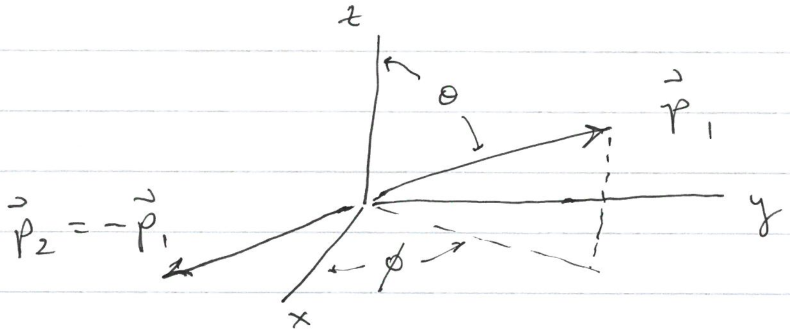
But for $N_f = 2$ final-state ptcls,

momentum space has $3N_f = 6$ dimensions,

Energy/mom. conservation $\rightarrow$ 4 constraints

$\Rightarrow$ final state config. determined by $\underline{2}$ quantities

e.g. θ, ϕ of one of outgoing ptcls in cm frame



$\Rightarrow$ We can integrate

$$dLIPS = (2\pi)^4 \delta^4(P_f - P_i) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

over everything except θ, ϕ

Even if $|\mathcal{M}|^2$ depends on magnitudes

of $\vec{p}_1, \vec{p}_2$, $\delta^4(P_f - P_i)$ enforces

$\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$ and $E_i^2 = |\vec{p}|^2 + m_i^2$, $i = 1, 2$

$\Rightarrow$ can treat $|\mathcal{M}|^2$ as const. except wrt θ, ϕ

$E_1 + E_2 = E_{cm}$

in cm.

Integrate dLIPS over $\vec{p}_2 \rightarrow \vec{p}_2 = -\vec{p}_1 \equiv -\vec{p}$

from $\delta^3(\vec{p}_f - \vec{p}_i)$

$P_f = (E_1 + E_2, 0)$

$P_i = (E_{cm}, 0)$ $\qquad E_{cm}$ <u>set</u> by experiment

$$\rightarrow \quad dLIPS = \frac{(2\pi)^4 \delta(E_1 + E_2 - E_{cm})\, d^3\vec{p}}{(2\pi)^3\, 2E_1\, (2\pi)^3\, 2E_2}$$

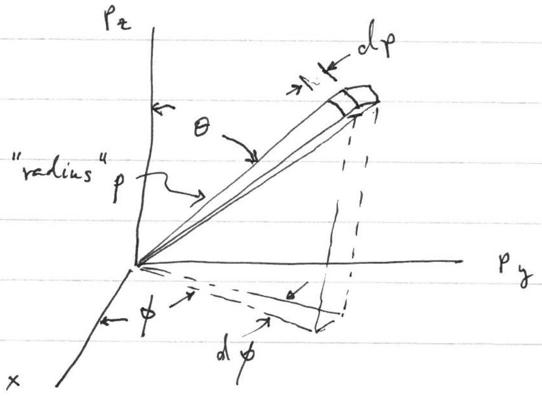
functions of $|\vec{p}|$

Final state now determined by $\vec{p}$, and

$\delta(E_1 + E_2 - E_{cm})$ enforces overall energy cons.

In spherical polar coord.

$$d^3\vec{p} = p^2\, dp\, \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$



Let $p = |\vec{p}|$

$$\delta(E_1 + E_2 - E_{cm}) = \delta\Big(\underbrace{\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} - E_{cm}}_{f(p)}\Big)$$

Use $\delta(f(p)) = \sum_i' \dfrac{\delta(p - p_i)}{|f'(p_i)|}$ $\qquad$ sum over all roots p_i of $f(p)$

$f(p) = 0$ when $E_1(p) + E_2(p) = E_{cm}$, which is

when p = magnitude of momentum of (either) final state ptcl. (in c.m.)

After non-trivial calc., $f(p)=0$ gives

$$p = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \equiv p^* \quad \text{where} \quad s \equiv E_{cm}^2$$

and $\lambda(s, m_1^2, m_2^2) = (s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2$

Källén triangle function

(for $m_1, m_2 << E_{cm}$, $p^* \approx \frac{E_{cm}}{2}$)

$$\Rightarrow \delta(E_1(p) + E_2(p) - E_{cm}) = \frac{\delta(p-p^*)}{\left| \left.\frac{\partial E_1}{\partial p}\right|_{p^*} + \left.\frac{\partial E_2}{\partial p}\right|_{p^*} \right|}$$

$$= \frac{E_1(p^*)\, E_2(p^*)}{(E_1(p^*) + E_2(p^*))\, p^*}\, \delta(p - p^*)$$

$$\Rightarrow dLIPS = \frac{1}{(2\pi)^2} \frac{1}{2E_1\, 2E_2}\, p^2\, d\Omega\, \frac{E_1(p^*)\, E_2(p^*)\, \delta(p-p^*)}{(E_1(p^*) + E_2(p^*))\, p^*}$$

Integrate over p:

$$\boxed{\underset{\overset{\shortparallel}{d\Phi_2}}{dLIPS} = \frac{1}{16\pi^2} \frac{p^*}{E_{cm}}\, d\Omega}$$

Two-body phase space in c.m.

Solid $\measuredangle$ $\quad d\Omega = \sin\theta\, d\theta\, d\phi$

$$= d\cos\theta\, d\phi$$

$\longrightarrow 2\pi\, d\cos\theta$ if $|\mathcal{M}|^2$ indep. of ϕ

$\longrightarrow 4\pi$ if $|\mathcal{M}|^2$ indep. of θ and ϕ

Decay Rates

$N_i = 1$ initial state ptcl, mass M

$V^{N_i - 1}$ in Fermi's Golden Rule $\rightarrow 1$

In rest frame of ptcl, decay rate

$$dW \rightarrow d\Gamma = \frac{1}{2M} |\mathcal{M}|^2 (2\pi)^4 \delta^4(P_f - P_i) \prod_{k=1}^{N_f} \frac{d^3\vec{p}_k}{(2\pi)2E_k}$$

$$= \frac{|\mathcal{M}|^2 \, dLIPS}{2M}$$

If $M \rightarrow m_1, m_2 \quad (N_f = 2)$

$$d\Gamma = \frac{1}{32\pi^2 M^2} |\mathcal{M}|^2 p^* \, d\Omega$$

If $|\mathcal{M}|^2$ indep. of θ, ϕ (isotropic)

$$\int d\Omega \rightarrow 4\pi$$

$$\Rightarrow \boxed{\Gamma = \int d\Gamma = \frac{p^*}{8\pi M^2} |\mathcal{M}|^2}$$

Cross section for $\overbrace{A}^{\text{projectile}} + \overbrace{B}^{\text{target}} \longrightarrow f$

$$\underbrace{A + B}_{N_i = 2}$$

$$d\sigma = \frac{dW}{J} \quad \text{flux (of projectile } A\text{)}$$

$$J = \rho\, |\vec{v}_{rel}| = \frac{|\vec{v}_A - \vec{v}_B|}{V}$$

$\hookrightarrow \rho = \frac{1}{V}$ (one ptcl. per vol. V)

$$\Rightarrow d\sigma = \underbrace{\frac{|\mathcal{M}|^2 \, dLIPS}{4E_A E_B |\vec{v}_A - \vec{v}_B|}}_{\text{flux factor } F} \quad \text{factors of } V \text{ cancel}$$

In "lab" frame $\vec{p}_B = 0$, $E_B = m_B$

Use $\vec{v}_A = \frac{\vec{p}_A}{E_A}$, $\vec{v}_B = 0 \Rightarrow F_{lab} = 4|\vec{p}_A| m_B$

$d\sigma$, $|\mathcal{M}|^2$, $dLIPS$ Lorentz invariant*

* almost (ok along $\vec{p}_A$)

$\Rightarrow$ F should also be Lorentz invariant

$$\Rightarrow \text{take } F = 4\left((p_A \cdot p_B)^2 - m_A^2 m_B^2\right)^{1/2}$$

F now "manifestly" Lorentz invariant

Reduces to F_{lab} in lab frame (check)

In c.m. frame, $F = 4|\vec{p}_i^{\,*}| E_{cm}$

$$\vec{p}_i^{\,*} = \vec{p}_A = -\vec{p}_B$$

initial

Consider $A + B \to 1 + 2$ in c.m.

initial : $\vec{p}_A = -\vec{p}_B \equiv \vec{p}_i^{\,*}$

final : $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}_f^{\,*}$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|} |\mathcal{M}|^2} \qquad \text{(c.m.)}$$

If initial & final ptcls ultra-relativistic

$$\frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i|} \longrightarrow 1$$

& if $|\mathcal{M}|^2$ indep. of ϕ , $\int d\phi = 2\pi$

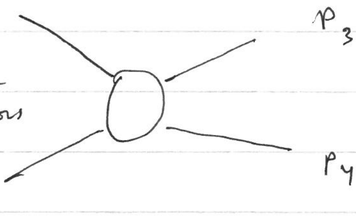
$$\Rightarrow \boxed{\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi E_{cm}^2} |\mathcal{M}|^2} \qquad \text{(c.m.)}$$

For arbitrary ref. frame, define Mandelstam var.

Lorentz invar.

$$\begin{cases} s = (p_1 + p_2)^2 = E_{cm}^2 \\ t = (p_1 - p_3)^2 \\ u = (p_1 - p_4)^2 \end{cases}$$

four-vectors



$t = m_1^2 + m_3^2 - 2E_1E_3 - 2\vec{p}_1 \cdot \vec{p}_3$

$\to -2E_1E_3(1 - \cos\theta_{13})$

$= -4E_1E_3 \sin^2\frac{\theta_{13}}{2}$

(relativistic lim.)

See notes: $$\frac{d\sigma}{dt} = \frac{|\mathcal{M}|^2}{16\pi\,\lambda(s, m_1^2, m_2^2)}$$

Spin averages, trace theorems

Amplitudes computed w/ Feynman rules specify spins of initial + final state ptcls.

But normally the incident ptcls are unpolarized (random mix of spin states) + detector usually does not distinguish between spin states.

$\Rightarrow$ average over initial spins

(and/or) sum over final spins

replace $|\mathcal{M}|^2 \longrightarrow \langle |\mathcal{M}|^2 \rangle$

This requires some tricks that depend on

$$\bar{u}(p,r)\, u(p,s) = 2m\,\delta_{rs}$$

$$\bar{v}(p,r)\, v(p,s) = -2m\,\delta_{rs}$$

$$\left.\begin{aligned} \sum_{s=1,2} u(p,s)\,\bar{u}(p,s) &= \not{p} + m \\ \sum_{s=1,2} v(p,s)\,\bar{v}(p,s) &= \not{p} - m \end{aligned}\right\}$$

Completeness theorem (see Notes Sec. 4.5)

Casimir's trick

some collection of γ matrices, e.g., γ^μ

Suppose $\mathcal{M} \sim \bar{u}_1 \Gamma u_2$

u_2 means $u(p_2, s_2)$

Then $|\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma u_2][\bar{u}_1 \Gamma u_2]^*$

$= [\bar{u}_1 \Gamma u_2][u_1^\dagger \gamma^0 \Gamma u_2]^\dagger$ ← same as $*$ since [ ] scalar

$= [\bar{u}_1 \Gamma u_2][u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1]$ ← since $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$

$= [\bar{u}_1 \Gamma u_2][u_2^\dagger \underbrace{\gamma^0\gamma^0}_{1} \Gamma^\dagger \gamma^0 u_1]$ ← since $\gamma^{0\dagger} = \gamma^0$

$= [\bar{u}_1 \Gamma u_2][\bar{u}_2 \bar{\Gamma} u_1]$

$\bar{u}_2 = u_2^\dagger \gamma^0$, $\bar{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0$

One can show e.g. $\overline{\gamma^\mu} = \gamma^\mu$, $\overline{\gamma^5} = -\gamma^5$,

$\overline{\not a \not b \not c \cdots \not p} = \not p \cdots \not c \not b \not a$ $\qquad \overline{\gamma^\mu \gamma^5} = \gamma^\mu \gamma^5$

Now use completeness relation

$$\sum_{s_i=1}^{2} u(p_i, s_i)\bar{u}(p_i, s_i) = \not p_i + m_i$$

$$\Rightarrow \sum_{s_2} |\mathcal{M}|^2 \sim \sum_{s_2} [\bar{u}_1 \Gamma \underbrace{u_2][\bar{u}_2}_{\not p_2 + m_2} \bar{\Gamma} u_1]$$

$$= \bar{u}_1 \underbrace{\Gamma (\not p_2 + m_2) \bar{\Gamma}}_{\equiv Q} u_1$$

Now use

$$\bar{u}_1 Q u_1 = \sum_{a,b=1}^{4}{}' (\bar{u}_1)_a Q_{ab} (u_1)_b$$

$$= \sum_{a,b=1}^{4}{}' Q_{ab} (\overbrace{u_1 \bar{u}_1}^{4\times 4 \text{ matrix}})_{ba}$$

$$= \sum_a{}' \left[Q u_1 \bar{u}_1 \right]_{aa}$$

$$= \mathrm{Tr}\left[Q u_1 \bar{u}_1 \right]$$

Apply again completeness wrt s_1 : $\sum_{s_1}' u_1 \bar{u}_1 = \not{p}_1 + m_1$

$$\sum_{s_1}{}' \bar{u}_1 Q u_1 = \mathrm{Tr}\left[Q (\not{p}_1 + m) \right]$$

$$\Rightarrow \sum_{s_1,s_2}{}' |\mathcal{M}|^2 = \mathrm{Tr}\left[\Gamma (\not{p}_2 + m_2) \bar{\Gamma} (\not{p}_1 + m_1) \right]$$

ok to cyclic permute

i.e. $$\boxed{\sum_{s_1,s_2}{}' [\bar{u}_1 \Gamma_A u_2][\bar{u}_1 \Gamma_B u_2]^* = \mathrm{Tr}\left[(\not{p}_1 + m_1) \Gamma_A (\not{p}_2 + m_2) \bar{\Gamma}_B \right]}$$

"Casimir's trick"

If sum contains $\bar{v}_i$, v_i, $m \to -m_i$

(since $\sum_{s_i=1,2}' v_i(p_i,s_i)\bar{v}_i(p_i,s_i) = \not{p}_i - m_i$

If $E_{cm} \gg m_1, m_2$

result $\to \mathrm{Tr}\left[\not{p}_1 \Gamma_A \not{p}_2 \bar{\Gamma}_B \right]$

Some trace theorems (cf. Aitcheson + Hey app. J.2)

$$\mathrm{Tr}(I_4) = 4$$

$$\mathrm{Tr}(\text{any odd \# of } \gamma \text{ matrices}) = 0$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = \tfrac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

$$= \tfrac{1}{2}\mathrm{Tr}(2g^{\mu\nu} I_4)$$

$$= g^{\mu\nu}\mathrm{Tr}(I_4) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\sigma})$$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu \not{p} \gamma^\nu \not{k}) = p_\lambda k_\sigma \mathrm{Tr}(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma)$$

$$= 4p_\lambda k_\sigma(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\lambda\nu} - g^{\mu\nu}g^{\lambda\sigma})$$

$$= 4[p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(p\cdot k)]$$

similarly $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) + (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d)]$

Also for $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ we will need in Ch 8.

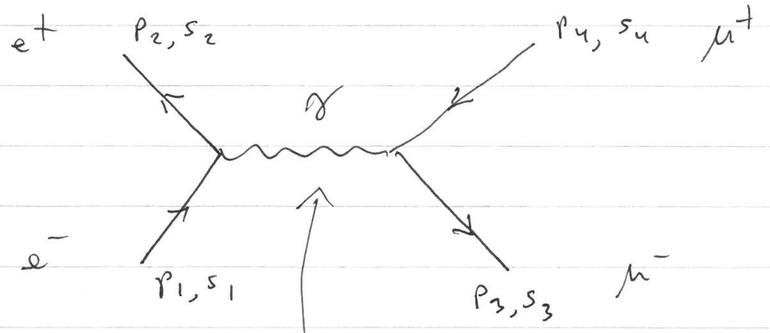
$$\mathrm{Tr}(\gamma^5) = 0$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\varepsilon^{\mu\nu\rho\sigma}$$

↑ antisymmetric tensor w/ $\varepsilon^{0123} = 1$

⚠ in some refs. $\varepsilon_{0123} = 1$, $\varepsilon^{0123} = -1$

Back to $e^+e^- \to \mu^+\mu^-$



Assume QED
(only γ, no Z)

+ $E_{cm} \gg m_e, m_\mu$

$$q = p_1 + p_2 = p_3 + p_4 \,, \quad q^2 = E_{cm}^2 \equiv s$$

We found

$$\mathcal{M} = \frac{i e^2}{s}\left[\bar{v}(p_2,s_2)\gamma_\mu u(p_1,s_1)\right]\left[\bar{u}(p_3,s_3)\gamma^\mu v(p_4,s_4)\right]$$

$$\Rightarrow |\mathcal{M}|^2 = \frac{e^4}{s^2}\left[\bar{v}_2\gamma_\mu u_1\right]\left[\bar{v}_2\gamma_\nu u_1\right]^*\left[\bar{u}_3\gamma^\mu v_4\right]\left[\bar{u}_3\gamma^\nu\right]^*$$

$$\Rightarrow \langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{\substack{s_1,s_2\\ s_3,s_4}}' |\mathcal{M}|^2 = \frac{e^4}{4s^2}\operatorname{Tr}\left[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu\right]\cdot\operatorname{Tr}\left[\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu\right]$$

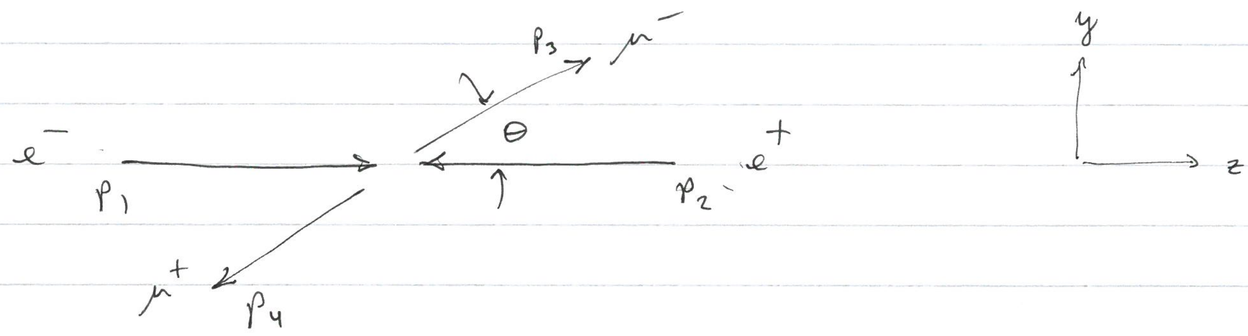
$$= \frac{4e^4}{s^2}\left[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot p_2)\right]\left[p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - g^{\mu\nu}(p_3\cdot p_4)\right]$$

$$= \frac{4e^4}{s^2}\left[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + \underbrace{(\overbrace{g_{\mu\nu}g^{\mu\nu}}^{4} - 4)}_{0}(p_1\cdot p_2)(p_3\cdot p_4)\right]$$

Use $\alpha = \dfrac{e^2}{4\pi} \simeq \dfrac{1}{137}$

$$\Rightarrow \; = \frac{128\pi^2\alpha^2}{s^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right]$$

Go to c.m. frame



$p_1 = (E, 0, 0, E)$ ∼ $E = E_{beam} = \dfrac{E_{cm}}{2}$

$p_2 = (E, 0, 0, -E)$

$p_3 = (E, 0, E\sin\theta, E\cos\theta)$ ∼ suppose event in (y,z) plane

$p_4 = (E, 0, -E\sin\theta, -E\cos\theta)$

$$s = q^2 = (p_1 + p_2)^2 = \underbrace{p_1^2}_{m_1^2} + \underbrace{p_2^2}_{m_2^2} + 2p_1\cdot p_2 \approx 2p_1\cdot p_2$$

$$= E_{cm}^2 = 4E^2$$

$$\Rightarrow \quad p_1\cdot p_3 = p_2\cdot p_4 = E^2(1+\cos\theta)$$

$$p_1\cdot p_4 = p_2\cdot p_3 = E^2(1-\cos\theta)$$

$$p_1\cdot p_2 = p_3\cdot p_4 = 2E^2$$

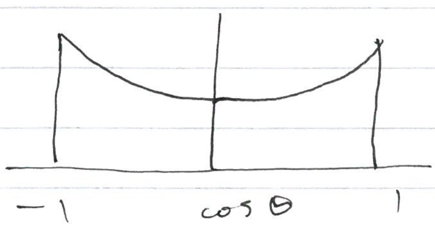
$$\Rightarrow \frac{1}{4}\sum_{spins}{}' |\mathcal{M}|^2 = \frac{128\pi^2\alpha^2}{(4E^2)^2}\left[E^4(1+\cos\theta)^2 + E^4(1-\cos\theta)^2\right]$$

$$= 16\pi^2\alpha^2(1+\cos^2\theta)$$

spin averaged

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}\times\frac{1}{4}\sum_{spins}{}'|\mathcal{M}|^2 = \frac{\alpha^2}{4E_{cm}^2}(1+\cos^2\theta)$$

or $$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2\pi}{2E_{cm}^2}(1+\cos^2\theta)}$$



In "experimentalists'" units

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \hbar^2 c^2}{4 E_{cm}^2} (1 + \cos^2\theta)$$

$\hbar c =$ 0.197 GeV·fm

Total cross section

$$\sigma = \int_{-1}^{1} \frac{d\sigma}{d\cos\theta}\, d\cos\theta = \frac{\pi\alpha^2}{2E_{cm}^2} \underbrace{\int_{-1}^{1} (1+x^2)\, dx}_{8/3}$$

$$= \frac{4\pi\alpha^2}{3E_{cm}^2}$$

$$\rightarrow \frac{4\pi\alpha^2 \hbar^2 c^2}{3 E_{cm}^2} = 0.103 \text{ nb at } E_{cm} = 29 \text{ GeV}$$

(PEP)

From 1984-86, the TPC/2γ Experiment at PEP (SLAC) collected $\underbrace{60 \text{ pb}^{-1}}$ of data

integrated luminosity $\int \mathcal{L}\, dt$

$\Rightarrow$ # of $e^+e^- \to \mu^+\mu^-$ events

$$N_{\mu\mu} = \sigma \cdot \int \mathcal{L}\, dt$$

$$= 0.103 \text{ nb} \times 60 \text{ pb}^{-1} \times \frac{1000 \text{ nb}^{-1}}{\text{pb}^{-1}}$$

$$= 6180 \text{ events}$$

(Extra – see notes)

Completeness relation for Dirac spinors

Let $A(p) = \sum_s{}' u(p,s)\,\bar{u}(p,s)$

Mult on right by $u(p,r)$

$$A(p)\,u(p,r) = \sum_s{}' u(p,s)\underbrace{\bar{u}(p,s)\,u(p,r)}_{2m\delta_{rs}}$$

$$= u(p,r)\cdot 2m$$

But $(\not{p} - m)\,u(p,r) = 0$

$\Rightarrow \quad (A - 2m)\,u = (\not{p} - m)\,u = 0$

$\Rightarrow \quad A = \not{p} + m$

$\Rightarrow \quad \sum_s{}' u(p,s)\,\bar{u}(p,s) = \not{p} + m$

Similarly $\quad \sum_s{}' v(p,s)\,\bar{v}(p,s) = \not{p} - m$

(because $\bar{v}(p,s)\,v(p,r) = -2m\delta_{rs}$)