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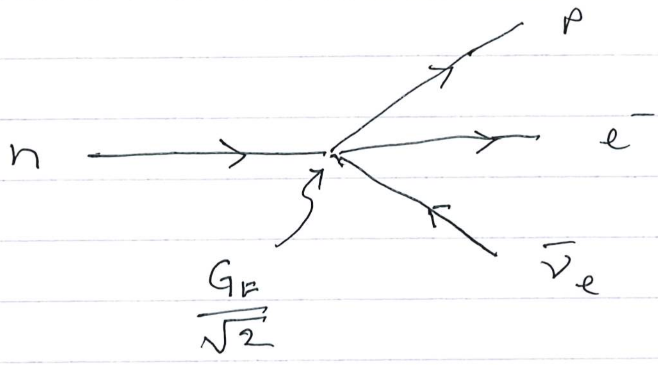
7.11.25

PH4442 Adv. Ptcl. Physics Week 6

Weak Interactions ($\rightarrow$ V-A theory)

Review main ideas (PH3520 Ch. 9)

1934 Fermi's theory of β decay



four-fermion vertex

$$G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$$

In analogy w/ QED, suppose current-current interaction

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} J_\mu^{had}\, j^\mu_{lep}$$

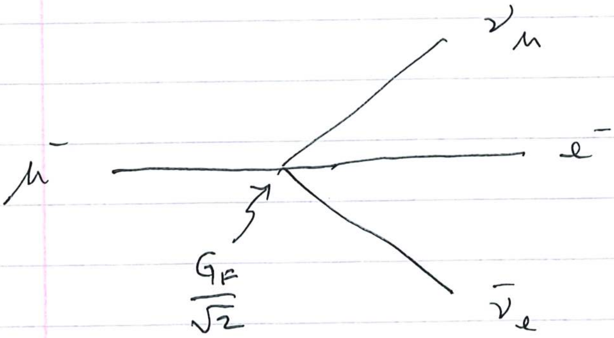
$$= \frac{G_F}{\sqrt{2}} \underbrace{\left[\bar{u}_p \gamma_\mu u_n \right]}_{\Delta Q = +1} \underbrace{\left[\bar{u}_e \gamma^\mu v_\nu \right]}_{\Delta Q = -1}$$

"current" is charge changing

Initially guess J_μ^{had}, j^μ_{lep} are Lorentz vectors

$\rightarrow \mathcal{M}$ is Lorentz scalar (conserves parity)

Can use Fermi's theory also for



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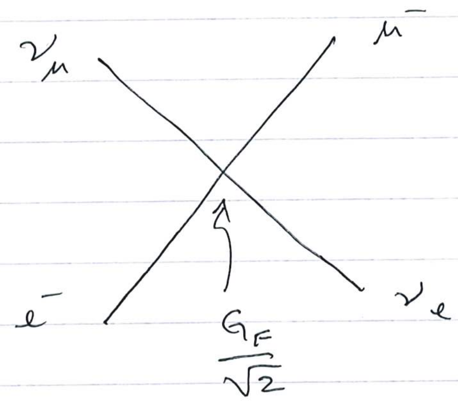


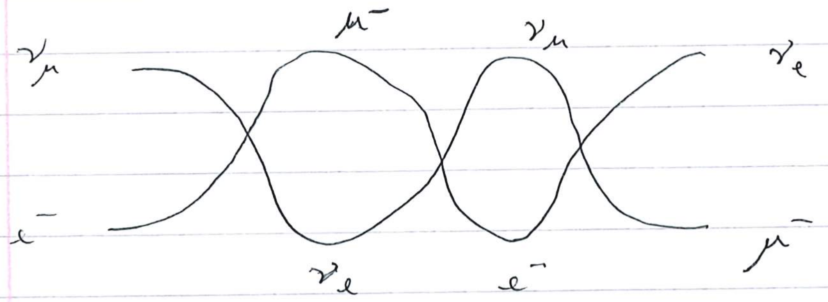
$\mu^- \to e^- \bar{\nu}_e \nu_\mu$ $\qquad\qquad$ $\nu_\mu e^- \to \mu^- \nu_e$

Dimensional arg. $\sigma(\nu_\mu e^- \to \mu^- \nu_e) \sim G_F^2 E_{cm}^2$

violates unitarity limit (Probability > 1)

at $E_{cm} \simeq 300$ GeV

Need higher order diagrams? Include e.g.



$\longrightarrow \infty$!?

QED also has ∞s in higher order terms,

but can be "renormalised" ($\to$ finite predictions)

Doesn't work for Fermi's theory.

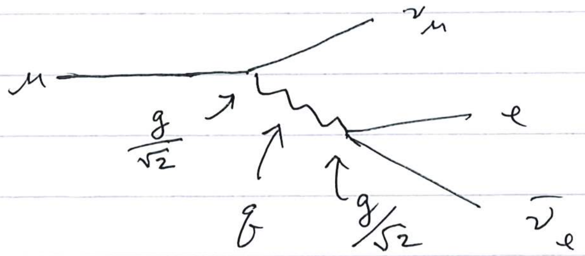
Analogy w/ γ of QED : allow interaction of μ, e, ν_e, ν_μ to be mediated by an intermediate vector (spin-1) boson (IVB)

↑ todays $W^\pm$

To make rate <u>weak</u>, IVB massive

$$\frac{-i g^{\mu\nu}}{q^2} \longrightarrow \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_W^2}\right)}{q^2 - M_W^2} \quad (+i\varepsilon)$$

E.g. in μ decay



q^2 = inv. mass of e, $\bar{\nu}_e$

max when $E_{\nu_\mu} \to 0 \Rightarrow q^2 = m_\mu^2$, $m_\mu = 0.1057$ GeV

So $q^2 \ll M_W^2$ ($M_W = 80.4$ GeV)

$$\Rightarrow \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_W^2}\right)}{q^2 - M_W^2} \longrightarrow \frac{i g^{\mu\nu}}{M_W^2}$$

$$\Rightarrow \mathcal{M} \sim \frac{g^2}{M_W^2} \quad (q^2 \ll M_W^2)$$

$$\Rightarrow G_F = \frac{\sqrt{2}}{8}\frac{g^2}{M_W^2} \quad (\text{factors of } 8, \sqrt{2} \text{ historical})$$

[Now know $M_W = 80.4$ GeV, $g = 0.65$]

For $\nu_\mu e^- \to \mu^- \nu_e$ at high E_{cm},

$$\mathcal{M} \sim g^2 \frac{-i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_W^2}\right)}{q^2 - M_W^2} \sim \frac{g^2}{E_{cm}^2}$$

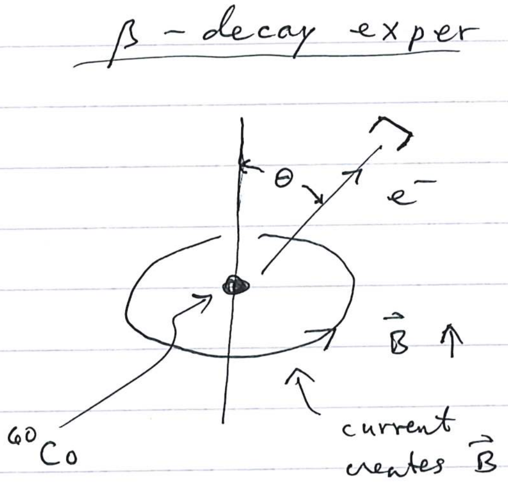
can neglect (not obvious)

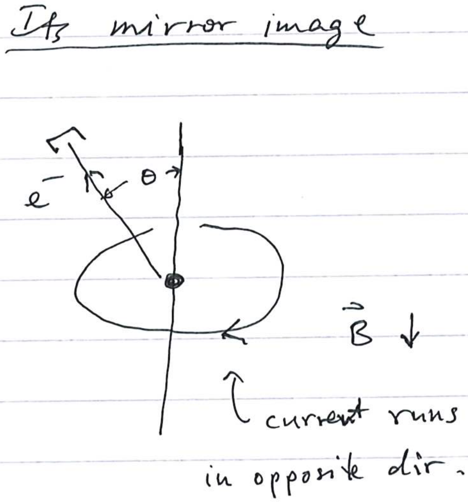
$q^2 \sim E_{cm}^2$

→ can (mostly) avoid unitarity bound

And, after including gauge symmetry (→ Z) one obtains a renormalizable theory (later)

Non-conservation of parity (PH3520 Sec. 9.3)





If parity conserved, $N(\theta) = N(\pi - \theta)$

∡ between e^- and $\vec{B}$

Exper. of C.S. Wu (1957): $\underline{N(\theta) \neq N(\pi - \theta)}$

⇒ parity <u>not</u> conserved

Theory & data agree if:

ν always "left-handed" (spin opposite to $\vec{p}$)

$\bar{\nu}$ " right- " (" parallel to $\vec{p}$)

Helicity and Chirality

$$\psi_- = v\, e^{ip\cdot x}$$

(Prelim.) For antiparticles, neg. energy states reinterpreted as antiptcls w/ $E, \vec{p} \to -E, -\vec{p}$

$\Rightarrow \quad \vec{L} = \vec{r} \times \vec{p} \longrightarrow -\vec{r} \times \vec{p}$

$\Rightarrow$ For $\vec{J} = \vec{L} + \vec{S}$ still to be conserved, need "physical" spin $\vec{S}^{(v)} = -\vec{S}$ for antiparticles.

Helicity $\equiv$ projection of (physical) spin along direction of motion

$$\hat{\lambda} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}, \qquad \vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$ or for Dirac ptcl.

E.g. consider ptcl. w/ $\vec{p} = \hat{z}\, p_z$, spinors are

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix}, \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For particle spinors u_1, u_2 w/ $\vec{p} = \hat{z}\,|\vec{p}\,|$

$$S_z u_1 = \frac{1}{2} u_1 \quad \rightarrow \quad \hat{\lambda} u_1 = \frac{1}{2} u_1 \qquad \text{pos. helicity}$$

$$S_z u_2 = -\frac{1}{2} u_2 \quad \rightarrow \quad \hat{\lambda} u_2 = -\frac{1}{2} u_2 \qquad \text{neg. helicity}$$

For antiparticle spinors v_1, v_2 w/ $\vec{p} = \hat{z}\,|\vec{p}\,|$, physical spin found to be

$$S_z^{(v)} v_1 = -S_z v_1 = \frac{1}{2} v_1$$

$$S_z^{(v)} v_2 = -S_z v_2 = -\frac{1}{2} v_2$$

Corresponding physical helicity op. for v_1, v_2

$$\hat{\lambda}^{(v)} = -\hat{\lambda} = -\frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}\,|}$$

For (still) $\vec{p} = \hat{z}\,|\vec{p}\,|$

$$\hat{\lambda}^{(v)} v_1 = \frac{1}{2} v_1 \qquad \text{pos. helicity}$$

$$\hat{\lambda}^{(v)} v_2 = -\frac{1}{2} v_2 \qquad \text{neg. helicity}$$

Chirality (handedness) χερι

Recall $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\left((\gamma^5)^2 = 1,\ (\gamma^5)^\dagger = \gamma^5,\ \{\gamma^5, \gamma^\mu\} = 0\right)$

Define right-, left-chirality projection ops

$$P_L = \frac{1}{2}(1 - \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$P_R = \frac{1}{2}(1 + \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

here means I_4

P_R, P_L have usual properties of projection ops.:

$$P_L + P_R = 1$$

$$P_L^2 = P_L \quad , \quad P_R^2 = P_R$$

$$P_L P_R = P_R P_L = 0$$

For Dirac spinors u, v

$$\left.\begin{aligned} u_L &= P_L u \quad , & v_L &= P_R v \\ u_R &= P_R u & v_R &= P_L v \end{aligned}\right\} \triangle!$$

$$u = (P_L + P_R)\, u = u_L + u_R$$

$$v = (P_L + P_R)\, v = v_R + v_L$$

↓ slides

Apply P_R, P_L to u_1 w/ $\vec{p} = \hat{z}\, |\vec{p}|$

$$P_R u_1 = \frac{1}{2}\sqrt{E+m}\left(1 + \frac{p_z}{E+m}\right)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} , \quad P_L u_1 = \frac{1}{2}\sqrt{E+m}\left(1 - \frac{p_z}{E+m}\right)\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

i.e. for $\vec{p} = \hat{z}\,|\vec{p}|$ and $E \gg m$ (rel. limit)

$$\left.\begin{aligned} P_R u_1 &\to u_1 & P_R v_1 &\to 0 \\ P_L u_1 &\to 0 & P_L v_1 &\to v_1 \end{aligned}\right\} \text{spin up (along } \hat{z}\text{)}$$

$$\left.\begin{aligned} P_R u_2 &\to 0 & P_R v_2 &\to v_2 \\ P_L u_2 &\to u_2 & P_L v_2 &\to 0 \end{aligned}\right\} \text{spin down (rel. to } \hat{z}\text{)}$$

Direction of $\hat{z}$ arbitrary $\Rightarrow$ in rel. limit, $\triangle!$

pos. helicity ptcl = right chiral $u_R = P_R u$ pos hel. antiptcl. = right-chiral $v_R = P_L v$

neg. " " = left " $u_L = P_L u$ neg. " " = left-chiral $v_L = P_R v$

I.e. in rel. limit, right-chiral/handed ↔ pos. hel.
for ptcls and antiptcls: left-chiral ↔ neg hel.

V − A theory

Fermi's original theory : $\mathcal{M} \sim J_\mu^{had} \, j^\mu_{lep}$ ← β decay

After $\cancel{P}$ (1957) need to modify, still should be inv. under boosts, rotation, translation

⇒ construct currents out of the bilinear covariants

S	scalar	$\bar\psi \psi$
P	pseudoscalar	$\bar\psi \gamma^5 \psi$
V	vector	$\bar\psi \gamma^\mu \psi$
A	axial vector	$\bar\psi \gamma^\mu \gamma^5 \psi$
T	tensor	$\bar\psi \sigma^{\mu\nu} \psi$ ← $\frac{i}{2}[\gamma^\mu, \gamma^\nu]$

Agreement w/ experiment found for

$$J^\mu \sim V^\mu - A^\mu \qquad \text{(leptons + quarks)}$$

$$\text{e.g. } j^\mu_{lep} = \bar\psi_e \gamma^\mu \psi_\nu - \bar\psi_e \gamma^\mu \gamma^5 \psi_\nu$$

β-decay amplitude becomes

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \left[\bar u_p \gamma_\mu (1 - g_A \gamma^5) u_n\right]\left[\bar u_e \gamma^\mu (1-\gamma^5) v_\nu\right]$$

so that G_F stays same

because p = uud, n = ddu
quarks must obey spin/statistics
+ strong interaction
→ $g_A = 1.275$ (axial vector coupling)

For leptons, can write

$$V^\mu - A^\mu = 2\,\bar{\Psi}_e \gamma^\mu \underbrace{P_L}_{\frac{1}{2}(1-\gamma^5)} \Psi_\nu$$

$\Rightarrow$ only left-chiral component of ν enters through $u_L = P_L u$. (negative helicity ν)

For $\bar{\nu}$, only get $v_R = P_L v$

$\rightarrow$ right-chiral, positive helicity $\bar{\nu}$.

Leptonic current becomes

$$j^\mu_{Lep} = \bar{u}_e \gamma^\mu P_L v_\nu$$

$$= \bar{u}_e \gamma^\mu P_L^2 v_\nu \qquad P_L^2 = P_L$$

$$= \bar{u}_e \gamma^\mu \tfrac{1}{2}(1-\gamma^5) P_L v_\nu \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

$$= u_e^\dagger \gamma^0 \tfrac{1}{2}(1+\gamma^5)\gamma^\mu P_L v_\nu \qquad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$$

$$= u_e^\dagger \tfrac{1}{2}(1-\gamma^5)\gamma^0\gamma^\mu P_L v_\nu \qquad \gamma^0\gamma^5 = -\gamma^5\gamma^0$$

$$= \left[\tfrac{1}{2}(1-\gamma^5) u_e\right]^\dagger \gamma^0 \gamma^\mu \underbrace{P_L v_\nu}_{(v_\nu)_R} \qquad \gamma^{5\dagger} = \gamma^5$$

$$= (\bar{u}_e)_L \gamma^\mu (v_\nu)_R$$

$\Rightarrow$ β-decay amplitude contains only left-chiral component of charged lepton (e^-).

Charged pion decay

Fact: $$\frac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)} = 1.23 \times 10^{-4}$$

i.e. very small, even though $e\nu$ final state has more available phase space (larger p^*)

Reason: conservation of angular momentum
& V−A structure of coupling.

Spin of $\pi^{\pm}$: $J_\pi = 0$ ($\measuredangle$ dist. of production + decay)

e, ν each spin $\frac{1}{2}$ ⇒ combine to $J=0$ or $J=1$

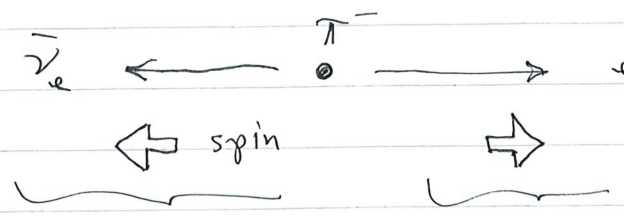
⇒ orbital $\measuredangle$ mom. L of $e\nu$ either 0 or 1

But, $L=1$ has "centrifugal barrier"

$$V_{cen} = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2}$$

Weak interact. short range, $r \sim \frac{1}{M_W} \approx 2.5 \times 10^{-18}$ m

⇒ $L=1$ suppressed, only $\underline{L=0}$



pos. hel. ↗ right-handed $\bar{\nu}_e$

by cons. of $\measuredangle$ mom. e^- has positive helicity e.g. if e^- along z, $u = u_1$

But coupling in weak int. only to $P_L u = u_L$

Recall $P_L u_1 = \frac{1}{2}(1-\gamma^5) u_1$

$$= \frac{1}{2}\sqrt{E+m}\underbrace{\left(1 - \frac{p}{E+m}\right)}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \text{ for } E \gg m$$

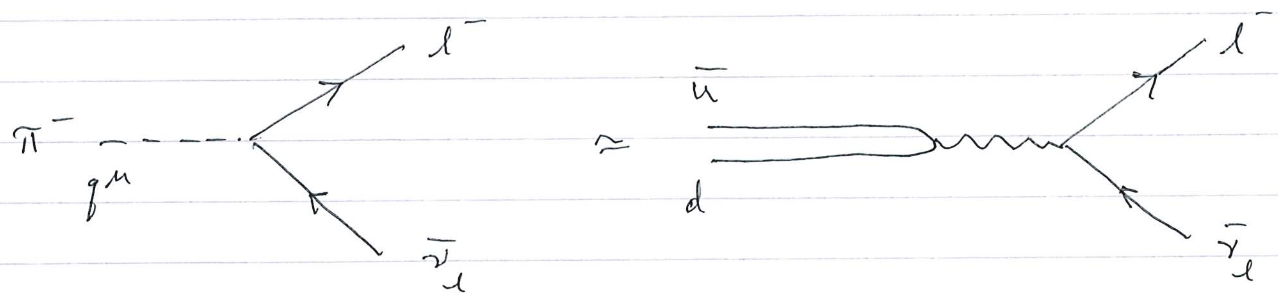
For $m_\pi = 140$ MeV

$m_\mu = 105.7$ MeV

$m_e = 0.5$ MeV

$$1 - \frac{p}{E+m} \simeq \begin{cases} 0.0073 & m = m_e \\ 0.86 & m = m_\mu \end{cases}$$

$\pi^- \to l^- \bar{\nu}_l$ amplitude $l = e, \mu$



$$\mathcal{M} = \frac{G_F}{\sqrt{2}} j^\pi_\mu \times \underbrace{\bar{u}_l \gamma^\mu (1-\gamma^5) v_\nu}_{\to V-A}$$

$\sim \bar{v}_u \gamma_\mu (1-\gamma^5) u_d$ if $\bar{u}, d \sim$ free

But in bound state, complicated by strong interact. effects

$\mathcal{M}$ must be invariant under boosts, rotations

$\Rightarrow j^\pi_\mu$ must be Lorentz vec. or axial vec.

Only relevant four-vec on which it can depend is q^μ

$\Rightarrow j^\pi_\mu = f_\pi q_\mu$ (Find $f_\pi \simeq 130$ MeV)

In rest frame of π, $q_\mu = (m_\pi, 0, 0, 0)$

$\Rightarrow \mathcal{M}$ only has $\mu = 0$ component of currents

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} f_\pi m_\pi \bar{u}_\ell \gamma^0 (1-\gamma^5) v_\nu$$

Assign coord. sys. w/ ℓ^- along $\hat{z}$ axis

ℓ^- $\quad \vec{p} = \hat{z}\, p$

$\bar{\nu}_\ell$ $\quad \vec{k} = -\vec{p} = -\hat{z}\, p$

For $\bar{\nu}_\ell$, spin must also be in $-\hat{z}$ dir (right-handed $\bar{\nu}$)

$\uparrow$ pos. hel.

$\Rightarrow \bar{\nu}$ spinor is $v_2(p_z = -k) = \sqrt{k} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

By cons. of $\not{J}$ mom., spin of ℓ^- in $+\hat{z} \Rightarrow \underline{u_1}$

$\Rightarrow$ also positive helicity

If u_2 included, makes no contrib. to $\mathcal{M}$

$$\bar{u}_2 \gamma^0 (1-\gamma^5) v_2 = u_2^\dagger \underbrace{\gamma^0\gamma^0}_{1} \underbrace{(1-\gamma^5) v_2}_{2 v_2} = 2 u_2^\dagger v_2 = 0$$

$\Rightarrow$ no u_2. Form of $\mathcal{M}$ ensures $\not{J}$ mom. conserved.

$$\Rightarrow \mathcal{M} = \frac{G_F}{\sqrt{2}} f_\pi m_\pi \underbrace{\bar{u}_{1,l}\gamma^0}_{u^\dagger_{1,l}} \underbrace{(1-\gamma^5)v_{2,\nu}}_{2v_{2,\nu}}$$

$$= \frac{2G_F}{\sqrt{2}} f_\pi m_\pi u^\dagger_{1,l} v_{2,\nu}$$

$$= \frac{2G_F}{\sqrt{2}} f_\pi m_\pi \sqrt{E+m_l}\left(1, 0, \frac{p}{E+m_l}, 0\right)\sqrt{k}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$$

$$= \frac{2G_F}{\sqrt{2}} f_\pi m_\pi \sqrt{p}\left(\sqrt{E-m_l} - \sqrt{E+m_l}\right)$$ ← used $k=p$

$$\Rightarrow |\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\pi^2 E(E-p)$$

$$= 2G_F^2 f_\pi^2 m_l^2 (m_\pi^2 - m_l^2)^2$$ ← indep. of θ

For isotropic decay, use (for c.m.)

$$\Gamma = \frac{p}{8\pi m_\pi^2}|\mathcal{M}|^2$$ $p = p^*$ = momentum of l^- (or $\bar{\nu}$) in c.m.

$$\Rightarrow \boxed{\Gamma = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 m_l^2 (m_\pi^2 - m_l^2)^2}, \quad l = e \text{ or } \mu$$

$$\Rightarrow \frac{\Gamma(\pi^- \to e^-\bar{\nu})}{\Gamma(\pi^- \to \mu^-\bar{\nu})} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 1.28 \times 10^{-4}$$

$\approx$ exper. value (diff. can be accounted through higher-order corrections)

$\Rightarrow$ big success for V - A theory!

Bulk of effect (small rate for e^-)

driven by $|\mathcal{M}|^2$, not by phase-space

factor $\propto$ $p_\ell = \frac{1}{2m_\pi}(m_\pi^2 - m_\ell^2)$

$$\frac{p_e}{p_\mu} = 2.4 \quad \text{(more phase space for } \pi \to e^- \bar{\nu}\text{)}$$

Other possibilities for coupling considered

S (scalar) $\quad \mathcal{M} \sim \bar{u}_e v_\nu$

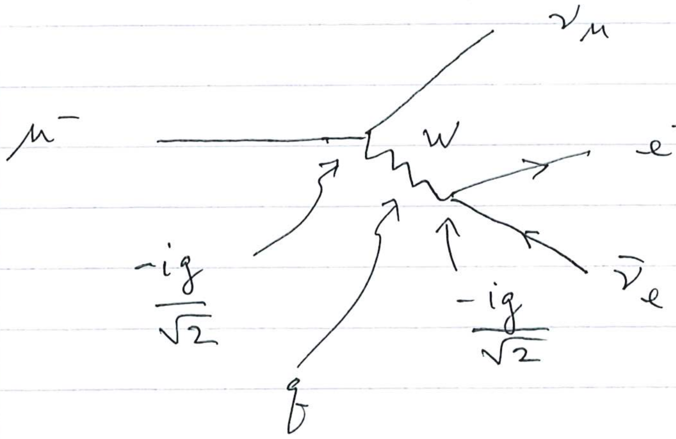
P (pseudoscalar) $\quad \mathcal{M} \sim \bar{u}_e \gamma^5 v_\nu$

Amplitude for positive helicity e^- not suppressed,

$$\frac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)} \approx \underset{\substack{\uparrow \\ \text{S or P}}}{5.5} \neq \text{experiment}$$

→ slides

Muon decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$



$$\mathcal{M} = -\frac{ig}{\sqrt{2}}\left[\bar{u}(\nu_\mu)\gamma_\mu \tfrac{1}{2}(1-\gamma^5)u(\mu)\right]\left(\frac{-ig^{\mu\nu} + \frac{iq^\mu q^\nu}{M_W^2}}{q^2 - M_W^2}\right)\left(-\frac{ig}{\sqrt{2}}\right)$$

$$\times\left[\bar{u}(e)\gamma_\nu \tfrac{1}{2}(1-\gamma^5)v(\bar{\nu}_e)\right]$$

$$= \frac{G_F}{\sqrt{2}} J_\mu(\mu) J^\mu(e)$$ long calc. because of 3-body final state

unpolarized μ

$$\hookrightarrow \frac{d\Gamma}{d\Omega\, dx} = \frac{G_F^2 m_\mu^5}{192\pi^4} x^2\left[3(1-x) + \frac{2}{3}\rho(4x-3) + 6\eta \frac{m_e}{m_\mu}\frac{1-x}{x}\right]$$

$$x = \frac{E_e}{E_{e,max}} = \frac{2E_e}{m_\mu}$$

For V−A theory, $\underbrace{\rho = \frac{3}{4},\ \eta = 0}_{\text{Michel parameters}}$

$$\Gamma = \int \frac{d\Gamma}{dx\, d\Omega} dx\, d\Omega = \frac{G_F^2 m_\mu^5}{192\pi^3} = \Gamma(\mu^- \to e^- \nu\bar{\nu}) \approx \Gamma_{tot}$$

↑ only final state

$$\tau = \frac{1}{\Gamma_{tot}} = 2.2\,\mu s$$

↖ muon lifetime