PH4442 Advanced Particle Physics 2025/26
Lecture Week 2

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- Review of non-relativistic QM
- The Klein-Gordon equation
- Dirac u and v spinors

(other topics done on whiteboard)

Review of non-relativistic QM

Free particle of energy E and momentum p has wavefunction

$$\Psi(\boldsymbol{r},t) = \frac{1}{\sqrt{V}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

$$E = \hbar\omega\,,$$ (here reinsert $\hbar$)

$$\boldsymbol{p} = \hbar\boldsymbol{k}\,,$$

$$\lambda = 2\pi/|\boldsymbol{k}|$$ = de Broglie wavelength

Normalise in (large) volume V:

$$\int_V |\Psi(\boldsymbol{r},t)|^2\, d^3\boldsymbol{r} = 1$$

Factors of V cancel in final predictions, often just set $V = 1$.

Observables and operators

Observables correspond to operators, their eigenvalues are the possible outcomes of a measurement:

$$\begin{aligned}\hat{E}\Psi &= E\Psi = \hbar\omega\Psi \\ \hat{\boldsymbol{p}}\Psi &= \boldsymbol{p}\Psi = \hbar\boldsymbol{k}\Psi\end{aligned}$$

From plane-wave solution, the operators are therefore

$$\begin{aligned}\hat{E} &= i\hbar\frac{\partial}{\partial t} \, , \\ \hat{\boldsymbol{p}} &= -i\hbar\nabla \, .\end{aligned}$$

Non-relativistic Schrödinger equation

For a non-relativistic particle in a potential $V(\boldsymbol{r},t)$,

$$E = \frac{p^2}{2m} + V(\boldsymbol{r},t)$$

Substituting the operators for E and $\boldsymbol{p}$ and operating on Ψ gives the Schrödinger equation:

$$\underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r},t)\right)}_{=\,H}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

H ← the Hamiltonian

If V is time independent, separate variables: $\Psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})e^{-iEt/\hbar}$

Gives time independent Schrödinger eq.: $H\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})$

Probability continuity equation

Postulate: probability (density) to find particle: $\rho(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},\boldsymbol{t})|^2$

To find a continuity equation, consider Schrödinger eq. and its complex conjugate:

$$H\Psi = i\hbar\frac{\partial}{\partial t}\Psi\,, \qquad H^*\Psi^* = -i\hbar\frac{\partial}{\partial t}\Psi^* = H\Psi^*\,,$$

($H = H^*$ provided V is real)

Construct the quantity $$\Psi^* H\Psi - \Psi H\Psi^* = i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t} + \Psi\frac{\partial\Psi^*}{\partial t}\right)$$

Using the Hamiltonian $H = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r},t)$ this becomes

$$\frac{\partial}{\partial t}(\Psi^*\Psi) + \frac{\hbar}{2im}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = 0$$

(continuity equation)

Probability continuity equation (2)

Defining probability density and current as

$$\rho = \Psi^*\Psi \,,$$

$$\boldsymbol{j} = \frac{\hbar}{2im}\left[\Psi^*\nabla\Psi - (\nabla\Psi)^*\Psi\right]$$

the continuity equation becomes $$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0$$

Corresponding electric charge and current density for particle of charge q:

$$\rho_{\mathrm{e.m.}} = q\rho$$

$$\boldsymbol{j}_{\mathrm{e.m.}} = q\boldsymbol{j}$$

Probability density and current for free ptcl.

For a free particle, $\Psi(\boldsymbol{r},t) = \frac{1}{\sqrt{V}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, one finds

$$\rho = \Psi^*\Psi = \frac{1}{V},$$

$$\boldsymbol{j} = \frac{\hbar}{2im}\left[i\boldsymbol{k}\Psi^*\Psi - (-i\boldsymbol{k})\Psi^*\Psi\right] = \frac{\hbar\boldsymbol{k}}{2m}2\Psi^*\Psi = \frac{\boldsymbol{p}}{m}\rho = \boldsymbol{v}\rho$$

For a free particle, ρ and j are independent of space and time.

The Klein-Gordon equation

Start from relativistic relation of energy, momentum, mass

$$E^2 = |\boldsymbol{p}|^2 + m^2$$

and operators for energy and momentum

$$\hat{E} \;=\; i\frac{\partial}{\partial t}\,, \qquad \hat{\boldsymbol{p}} \;=\; -i\nabla\,,$$

acting on a scalar wave function $\varphi(x)$ gives

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi = 0\,, \qquad \text{or} \qquad (\partial_\mu \partial^\mu + m^2)\varphi = 0\,,$$

the Klein-Gordon eq. for a free particle of mass m.

Solutions to Klein-Gordon eq.

K-G eq. has both positive and negative energy solutions:

$$\varphi_+ = Ne^{-ip\cdot x} \quad \rightarrow \quad i\frac{\partial}{\partial t}\varphi_+ = +E\varphi_+ \,,$$

$$\varphi_- = Ne^{ip\cdot x} \quad \rightarrow \quad i\frac{\partial}{\partial t}\varphi_- = -E\varphi_- \,.$$

where "energy" means eigenvalue of $\hat{E} = i\frac{\partial}{\partial t}$

which can be positive or negative, and here

$$E = \sqrt{|\boldsymbol{p}|^2 + m^2} \geq 0$$

$$p \cdot x = Et - \boldsymbol{p} \cdot \boldsymbol{x}$$

Both φ_+ and φ_- needed to form complete set of basis functions.

K-G probability density, current

$|\varphi|^2$ is a Lorentz scalar but $|\varphi|^2\, dV$ is not (dV is Lorentz contracted).

$\longrightarrow |\varphi|^2$ cannot be a probability density,

One can define the probability density and current

$$j^\mu = (\rho, \boldsymbol{j}) = \frac{i}{2m}(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*)$$

(Factor $i/2m$ makes this coincide with Schrödinger $(\rho, \boldsymbol{j})$ in non-rel. limit.)

such that it is a four-vector and satisfies a continuity equation

$$\partial_\mu \boldsymbol{j}^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0$$

For the positive energy solutions φ_+, this gives

$$\rho = \frac{i}{2m}\left[(-iE)\varphi_+^* \varphi_+ - (+iE)\varphi_+^* \varphi_+\right] = \frac{E}{m}|N|^2$$

$$\boldsymbol{j} = -i\left[(i\boldsymbol{p})\varphi_+^* \varphi_+ - (-i\boldsymbol{p})\varphi_+^* \varphi_+\right] = \frac{\boldsymbol{p}}{m}|N|^2 \ .$$

Negative energy solutions $\longrightarrow$ negative probability

Using the negative energy solution φ_+ gives

$$\rho = i\left[(iE)\varphi_-^*\varphi_- - (-iE)\varphi_-^*\varphi_-\right] = -\frac{E}{m}|N|^2$$

i.e., a negative probability.

Klein, Gordon, Schrödinger give up on K-G equation.

Will resurrect later to describe antiparticles.

Change normalisation of current to

$$j^\mu = (\rho, \boldsymbol{j}) = i(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*) \ = 2p^\mu|N|^2$$

Dirac u and v spinors

Since the components of the free-particle Dirac equation solve the K-G eq., the solutions should be plane waves of the form

$$\begin{aligned} \psi_+(x) &= u(p)e^{-ip\cdot x} \,, \\ \psi_-(x) &= v(p)e^{ip\cdot x} \,. \end{aligned}$$

where the spinors $u(p)$ and $v(p)$ are four-component column vectors that can depend on the momentum but not x.

Substitute the positive energy solution $\psi_+(x)$ into the Dirac eq.:

$$(i\partial\!\!\!/ - m)u(p)e^{-ip\cdot x} = 0$$

$$\longrightarrow \quad (i(-ip\!\!\!/) - m)u(p)e^{-ip\cdot x} = 0$$

$$\longrightarrow \quad (p\!\!\!/ - m)u(p) = 0$$

Dirac u and v spinors (2)

With similar manipulation for $v(p)$ one finds

$$
\begin{aligned}
(\not{p} - m)u(p) &= 0 \\
(\not{p} + m)v(p) &= 0
\end{aligned}
$$

4 x 4 matrix

4-component column vector

For the (Dirac) adjoint spinors $\overline{u} = u^\dagger \gamma^0$ and $\overline{v} = v^\dagger \gamma^0$

$$
\begin{aligned}
\overline{u}(\not{p} - m) &= 0 \\
\overline{v}(\not{p} + m) &= 0
\end{aligned}
$$

4-component row vector

4 x 4 matrix

Rest-frame solutions for u and v

Consider free particle at rest: $p^\mu = (p^0, 0, 0, 0) = (m, \mathbf{0})$

Use $\not{p} = \gamma^\mu p_\mu = m\gamma^0$ in $(\not{p} - m)u(p) = 0$,

$$\rightarrow \quad (\gamma^0 - I)u(m, \mathbf{0}) = 0$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$u(m, \mathbf{0})$

Solved with arbitrary a, b with $c = d = 0$.

Rest-frame solutions for u and v (2)

We can take two linearly independent solutions for u to be

$$u_s(m, \mathbf{0}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = N \begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} , \qquad \varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

spin index
s = 1,2

normalisation
const.

2-component
Weyl spinors

In a similar way, find two solutions for v,

$$v_s(m, \mathbf{0}) = N \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} , \qquad \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Normalisation of φ, χ chosen such that $\varphi_r^\dagger \varphi_s = \chi_r^\dagger \chi_s = \delta_{rs}$

(does not yet fix normalisation of u_s, v_s).

Solutions for $u(p)$ and $v(p)$

Claim $u(p)$ and $v(p)$ found from rest-frame solutions using

$$u_s(p) = (\not{p} + m) u_s(m, \mathbf{0}) \ ,$$
$$v_s(p) = (\not{p} - m) v_s(m, \mathbf{0}) \ .$$

Proof (for u):

$$\begin{aligned} (\not{p} - m) u_s(p) &= (\not{p} - m)(\not{p} + m) u_s(m, \mathbf{0}) \\ &= (\not{p}\not{p} - m^2) u_s(m, \mathbf{0}) \\ &= (p^2 - m^2) u_s(m, \mathbf{0}) \\ &= 0 \ . \end{aligned}$$

← use $\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = p^2$

← since $p^2 = m^2$