PH4442 Advanced Particle Physics 2025/26
Lecture Week 3

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- Finish solving Dirac eq. (started last week)
- Interpretation of negative energy solutions of Dirac eq.
- Solving the Dirac equation in an E.M. potential (start)

Plan for week 3

Last week we derived the Dirac equation:

$$(i\partial\!\!\!/ - m)\psi(x) = qA\!\!\!/\psi(x)$$

and saw it has both positive and negative energy solutions.

This week:

- Finish solving Dirac equation (started last week)
- Transformation properties of Klein-Gordon and Dirac eqs.
- Interpretation of negative energy solutions $\rightarrow$ antiparticles
- Green functions, propagators (start this week)

Where this is heading:

- Feynman rules for amplitudes
- Predictions for decay rates and cross sections

Dirac u and v spinors

Since the components of the free-particle Dirac equation solve the K-G eq., the solutions should be plane waves of the form

$$\psi_+(x) = u(p)e^{-ip\cdot x} \,,$$
$$\psi_-(x) = v(p)e^{ip\cdot x} \,.$$

where the spinors $u(p)$ and $v(p)$ are four-component column vectors that can depend on the momentum but not x.

Substitute the positive energy solution $\psi_+(x)$ into the Dirac eq.:

$$(i\partial\!\!\!/ - m)u(p)e^{-ip\cdot x} = 0$$

$$\rightarrow \quad (i(-ip\!\!\!/) - m)u(p)e^{-ip\cdot x} = 0$$

$$\rightarrow \quad (p\!\!\!/ - m)u(p) = 0$$

Dirac u and v spinors (2)

With similar manipulation for $v(p)$ one finds

$$
\begin{aligned}
(\not{p} - m)u(p) &= 0 \\
(\not{p} + m)v(p) &= 0
\end{aligned}
$$

4 x 4 matrix

4-component column vector

For the (Dirac) adjoint spinors $\overline{u} = u^\dagger \gamma^0$ and $\overline{v} = v^\dagger \gamma^0$

$$
\begin{aligned}
\overline{u}(\not{p} - m) &= 0 \\
\overline{v}(\not{p} + m) &= 0
\end{aligned}
$$

4-component row vector

4 x 4 matrix

Rest-frame solutions for u and v

Consider free particle at rest: $p^\mu = (p^0, 0, 0, 0) = (m, \mathbf{0})$

Use $\not{p} = \gamma^\mu p_\mu = m\gamma^0$ in $(\not{p} - m)u(p) = 0$,

$$\rightarrow \quad (\gamma^0 - I)u(m, \mathbf{0}) = 0$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$u(m, \mathbf{0})$

Solved with arbitrary a, b with $c = d = 0$.

Rest-frame solutions for u and v (2)

We can take two linearly independent solutions for u to be

$$u_s(m, \mathbf{0}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = N \begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} , \qquad \varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

spin index s = 1,2

normalisation const.

2-component Weyl spinors

In a similar way, find two solutions for v,

$$v_s(m, \mathbf{0}) = N \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} , \qquad \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Normalisation of φ, χ chosen such that $\varphi_r^\dagger \varphi_s = \chi_r^\dagger \chi_s = \delta_{rs}$

(does not yet fix normalisation of u_s, v_s).

Solutions for $u(p)$ and $v(p)$

Claim $u(p)$ and $v(p)$ found from rest-frame solutions using

$$
\begin{aligned}
u_s(p) &= (\not{p} + m)u_s(m, \mathbf{0}) \, , \\
v_s(p) &= (\not{p} - m)v_s(m, \mathbf{0}) \, .
\end{aligned}
$$

Proof (for u):

$$
\begin{aligned}
(\not{p} - m)u_s(p) &= (\not{p} - m)(\not{p} + m)u_s(m, \mathbf{0}) \\
&= (\not{p}\not{p} - m^2)u_s(m, \mathbf{0}) \\
&= (p^2 - m^2)u_s(m, \mathbf{0}) \\
&= 0 \, .
\end{aligned}
$$

← use $\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = p^2$

← since $p^2 = m^2$

Explicit formulas for $u(p)$ and $v(p)$

Use $$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \quad i = 1, 2, 3 ,$$

to find that $$\not{p} = \gamma^\mu p_\mu = \begin{pmatrix} p^0 & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -p^0 \end{pmatrix}$$

The *unnormalised* solutions for u and v are therefore

$$u_s(p) = \begin{pmatrix} p^0 + m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -p^0 + m \end{pmatrix} \begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} = \begin{pmatrix} (p^0 + m)\varphi_s \\ (\boldsymbol{\sigma} \cdot \mathbf{p})\varphi_s \end{pmatrix}$$

$$v_s(p) = \begin{pmatrix} p^0 - m & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -p^0 - m \end{pmatrix} \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\sigma} \cdot \mathbf{p})\chi_s \\ (p^0 + m)\chi_s \end{pmatrix}$$

Normalisation of u and v

Want to interpret $\rho = \psi^\dagger \psi$ as a probability density

and this is 0th component of four-vector current $j^\mu = (\rho, \mathbf{j})$

Probability to find a particle in a given volume

$$P = \int d^3x\, j^0(x) = \int d^3x\, \psi^\dagger(x)\psi(x) = \begin{cases} \int d^3x\, u^\dagger u & \text{for } \psi_+ \\ \int d^3x\, v^\dagger v & \text{for } \psi_- \end{cases}$$

should be invariant under a Lorentz transformation. This will happen if $\rho = \psi^\dagger \psi$ transforms like 0th component of a four-vector, and the only four-vector on which it can depend is $p = (E, \boldsymbol{p})$.

Therefore take: $u^\dagger u \propto v^\dagger v \propto p^0$

Use convention: $u_r^\dagger u_s = v_r^\dagger v_s = 2E\delta_{rs}$ where $E = +\sqrt{|\mathbf{p}|^2 + m^2}$

From this find: $\bar{u}_r u_s = 2m\delta_{rs}\,, \quad \bar{v}_r v_s = -2m\delta_{rs}\,.$

Normalised $u(p)$ and $v(p)$

Using this normalisation we find $N = \sqrt{E+m}$ and therefore

$$u_1(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \Uparrow \qquad u_2(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \Downarrow$$

spin relative to z axis

$$v_1(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \Uparrow \qquad v_2(p) = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \Downarrow$$

Full solution to free-particle Dirac eq. includes exponential term,

$$\psi_+(x) = u(p)e^{-ip\cdot x}, \qquad \psi_-(x) = v(p)e^{ip\cdot x}.$$

Interpretation of negative energy solutions

Solutions to Dirac eq. have both positive and negative energy:

$$\psi_+ = u(p)e^{-ip\cdot x}$$

$$\psi_- = v(p)e^{ip\cdot x}$$

$$p\cdot x = Et - \boldsymbol{p}\cdot\boldsymbol{x}$$

$$E = \sqrt{|\boldsymbol{p}|^2 + m^2} \geq m$$

(here E always positive)

Energy means eigenvalue of $\hat{E} = i\frac{\partial}{\partial t}$

$$i\frac{\partial}{\partial t}\psi_+ = E\psi_+$$

$$i\frac{\partial}{\partial t}\psi_- = -E\psi_-$$

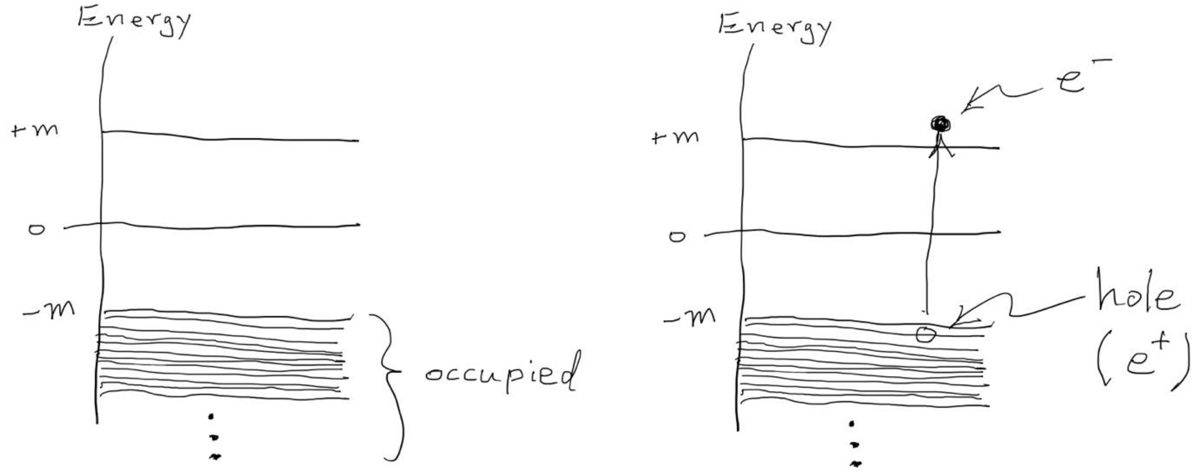
The Dirac sea (1930)

Dirac's first attempt was to say all negative energy states are filled.

Because of the Pauli exclusion principle, no particles could fall into the negative energy states.

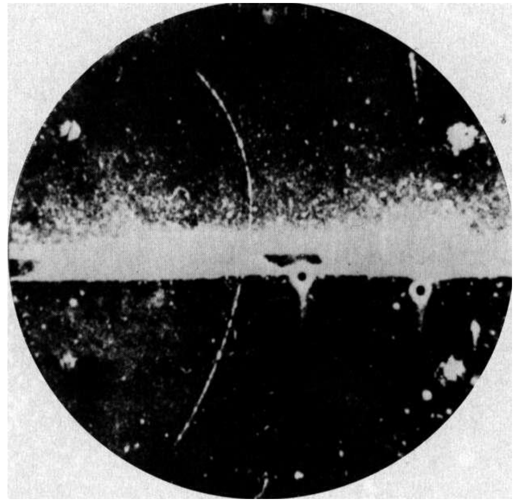
A negative energy state could be made vacant by adding an energy of at least $2m$ (e.g., collision of two photons).

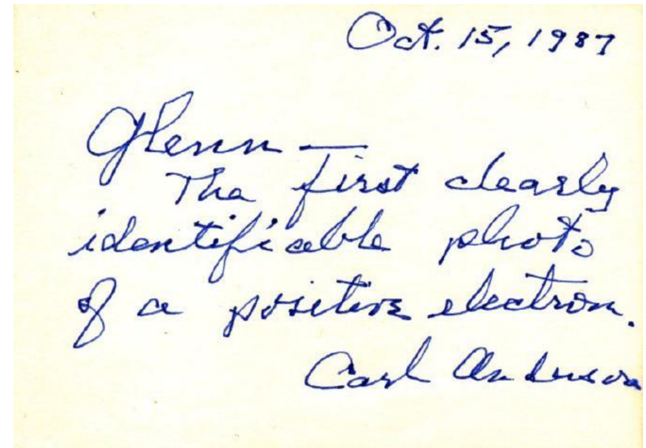
The vacated state or "hole" is an ***antiparticle*** (positive mass, opposite charge). Corresponds to pair production: $\gamma\gamma \to e^+e^-$.



Antimatter

Positron discovered (Anderson 1932).





But the Dirac sea could not be applied to bosons (since no Pauli principle).

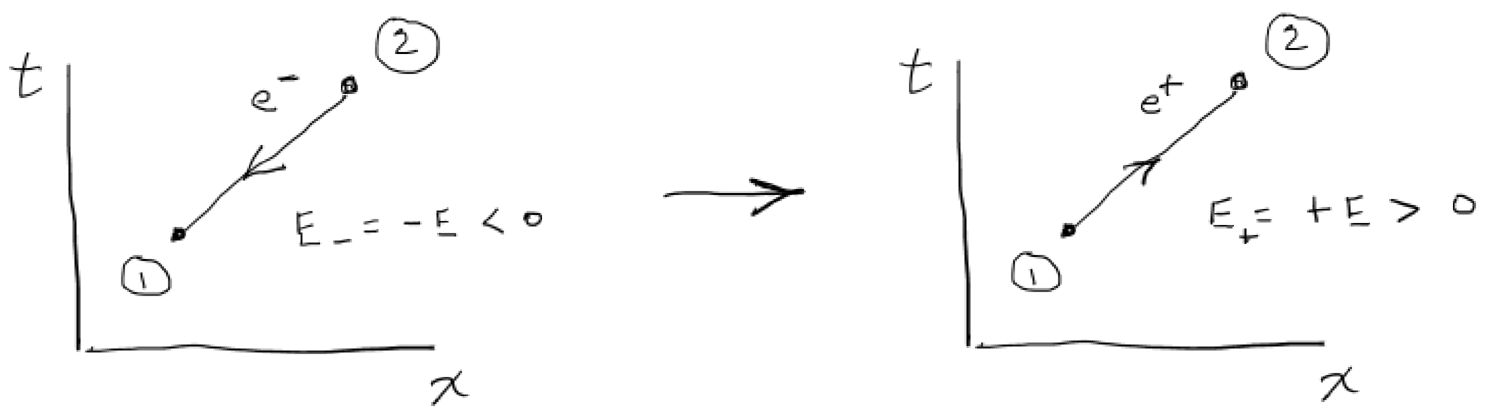
View abandoned in particle physics but its analog still used to describe electron-hole pairs in a semiconductor.

Feynman-Stückelberg interpretation

Stückelberg, Feynman interpret negative energy solution as propagating backwards in time, and equivalent to a positive energy particle moving forwards in time.

$$\psi_- \sim e^{iEt} = e^{i(-E)(-t)}$$

- Emission of an antiparticle with four-momentum p^μ is equivalent to absorption of a particle with $-p^\mu$.
- Absorption of an antiparticle with p^μ is equivalent to emission of a particle with $-p^\mu$.



Feynman-Stückelberg interpretation (2)

Feynman-Stückelberg interpretation gives consistent description of scattering processes including creation and annihilation of particles.

Dyson shows it is mathematically equivalent to Quantum Field Theory.

Usual view (today) is to take QFT as the more fundamental formulation of the theory.

Feynman-Stückelberg interpretation provides a fast route to Feynman rules for amplitudes.

Preparation for next week:

Review in the Lecture Notes:

Appendix B: Green functions

Appendix C: Complex contour integration

and/or the corresponding material from PH2130, PH3150.