PH4442 Advanced Particle Physics 2025/26
Lecture Week 4

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- Rules for Feynman Diagrams (Ch. 6)

For mathematical background see in the lecture notes:

- Appendix B: Introduction to Green Functions
- Appendix C: Introduction to Complex Integration

Plan for week 4

Finish from last week the Feynman-Stückelberg interpretation of negative energy solutions:

- Emission of an antiparticle with four-momentum p^μ is equivalent to absorption of a particle with $-p^\mu$.

- Absorption of an antiparticle with p^μ is equivalent to emission of a particle with $-p^\mu$.

This week we will use this to derive the rules for Feynman diagrams that will allow us to compute amplitudes for scattering and decay reactions. This will include:

- Green functions and propagators
- Amplitude for scattering by a potential
- Amplitude for electron-proton scattering
- Generalisation to Feynman rules for tree-level diagrams

Interpretation of negative energy solutions

Solutions to Dirac eq. have both positive and negative energy:

$$\psi_+ = u(p)e^{-ip\cdot x}$$

$$\psi_- = v(p)e^{ip\cdot x}$$

$$p \cdot x = Et - \boldsymbol{p} \cdot \boldsymbol{x}$$

$$E = \sqrt{|\boldsymbol{p}|^2 + m^2} \geq m$$

(here E always positive)

Energy means eigenvalue of $\hat{E} = i\frac{\partial}{\partial t}$

$$i\frac{\partial}{\partial t}\psi_+ = E\psi_+$$

$$i\frac{\partial}{\partial t}\psi_- = -E\psi_-$$

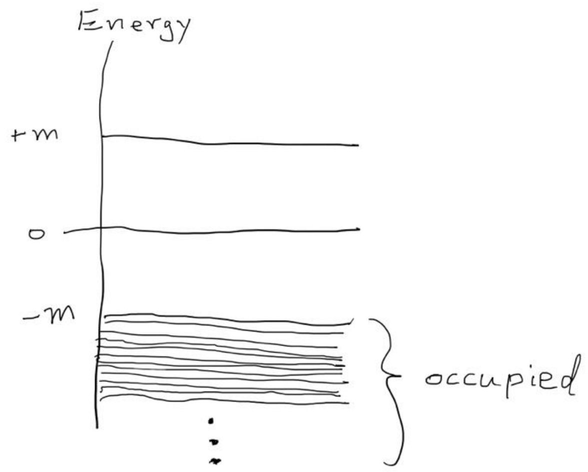
The Dirac sea (1930)

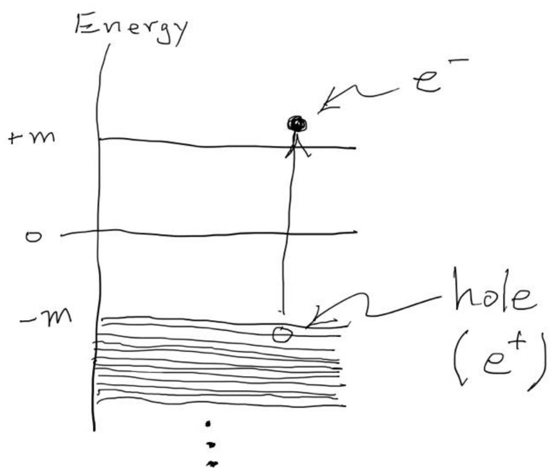
Dirac's first attempt was to say all negative energy states are filled.

Because of the Pauli exclusion principle, no particles could fall into the negative energy states.

A negative energy state could be made vacant by adding an energy of at least $2m$ (e.g., collision of two photons).

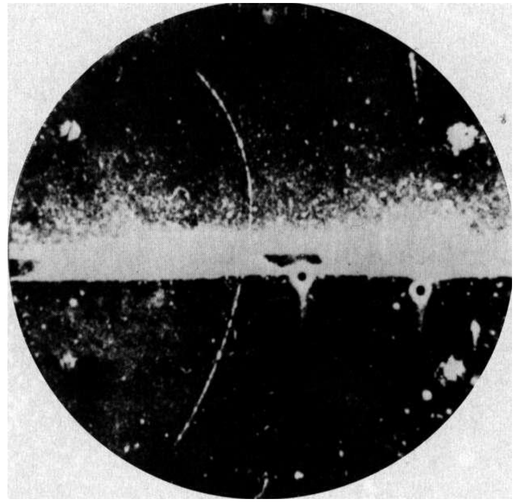
The vacated state or "hole" is an ***antiparticle*** (positive mass, opposite charge). Corresponds to pair production: $\gamma\gamma \rightarrow e^+e^-$.

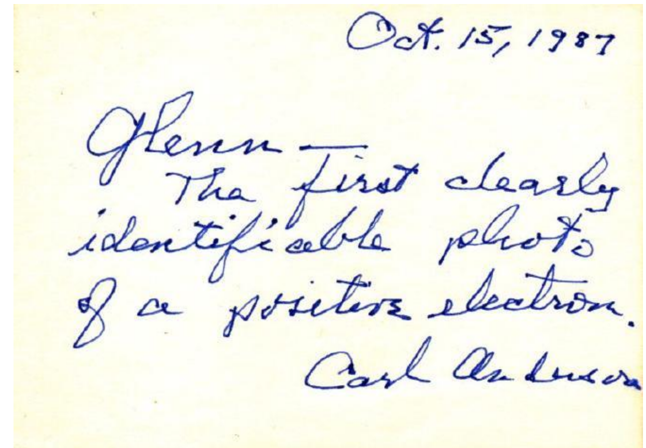




Antimatter

Positron discovered (Anderson 1932).





But the Dirac sea could not be applied to bosons (since no Pauli principle).

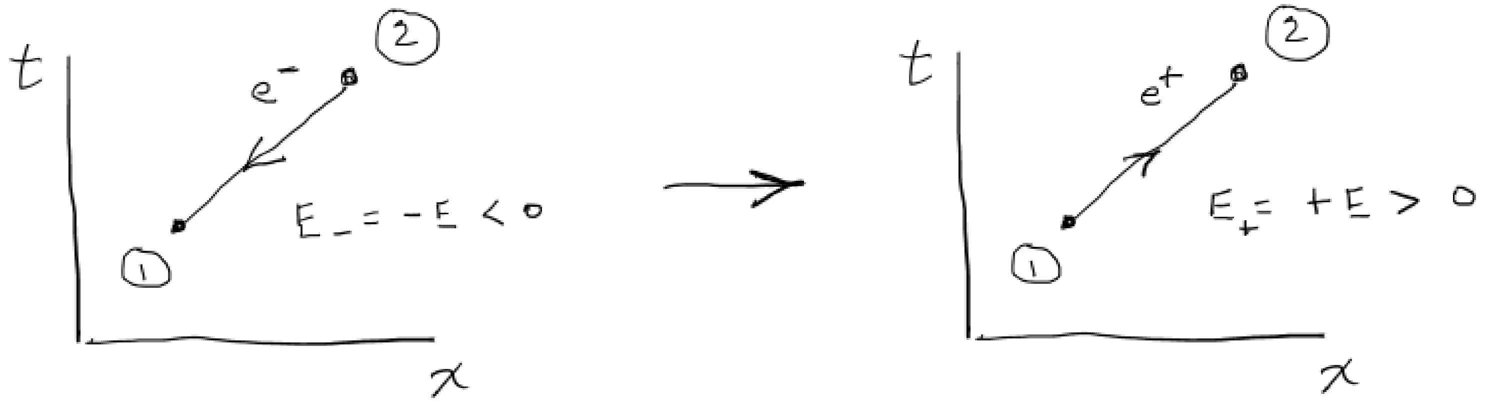
View abandoned in particle physics but its analog still used to describe electron-hole pairs in a semiconductor.

Feynman-Stückelberg interpretation

Stückelberg, Feynman interpret negative energy solution as propagating backwards in time, and equivalent to a positive energy particle moving forwards in time.

$$\psi_- \sim e^{iEt} = e^{i(-E)(-t)}$$

- Emission of an antiparticle with four-momentum p^μ is equivalent to absorption of a particle with $-p^\mu$.
- Absorption of an antiparticle with p^μ is equivalent to emission of a particle with $-p^\mu$.



Feynman-Stückelberg interpretation (2)

Feynman-Stückelberg interpretation gives consistent description of scattering processes including creation and annihilation of particles.

Dyson showed it is mathematically equivalent to Quantum Field Theory (F.J. Dyson, *The Radiation Theories of Tomonaga, Schwinger, and Feynman*, Phys. Rev. 75, 486 (1949))

Standard view (today) is to take QFT as the more fundamental formulation of the theory.

Feynman-Stückelberg interpretation provides a fast route to Feynman rules for amplitudes.

Integral representation of delta function

Take Fourier transform of $\delta(x - x')$

$$\hat{\delta}(k) = \int \delta(x - x') e^{-ikx} \, dx = e^{-ikx'}$$

Take inverse transform:

$$\delta(x - x') = \frac{1}{2\pi} \int e^{ik(x-x')} \, dk$$

Note either sign OK in exponent since $\delta(x - x') = \delta(x' - x)$

Often use 4-D version: $\delta^4(x - x') = \frac{1}{(2\pi)^4} \int e^{ik\cdot(x-x')} \, d^4k$

four-vectors

Propagator and time evolution (Sec. 6.4)

The Green function that solves $(i\partial\!\!\!/ - m)K(x - x') = \delta^4(x - x')$ is

$$K(x - x') = -\frac{i}{(2\pi)^3} \int d^3\boldsymbol{p} \exp\left[i\boldsymbol{p}\cdot(\boldsymbol{x} - \boldsymbol{x}') - iE(t - t')\right] \left(\frac{\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}{2E}\right), \qquad t > t' . \tag{1}$$

$$K(x - x') = -\frac{i}{(2\pi)^3} \int d^3\boldsymbol{p} \exp\left[i\boldsymbol{p}\cdot(\boldsymbol{x} - \boldsymbol{x}') - iE(t - t')\right] \left(\frac{-\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}{2E}\right), \qquad t < t' . \tag{2}$$

Consider free electron with $k = (k^0, \boldsymbol{k})$ with $k^0 > 0$. Wave function is

$$\phi_+(x') = \phi_+(t', \boldsymbol{x}') = u(k)\exp[-ik^0 t' + i\boldsymbol{k}\cdot\boldsymbol{x}'] \tag{3}$$

Claim: at a later time $t > t'$ the wave function is

$$\phi_+(x) = \phi_+(t, \boldsymbol{x}) = i\int d^3\boldsymbol{x}'\, K(x - x')\gamma^0\phi_+(t', \boldsymbol{x}') \tag{4}$$

Proof of integral formula for $\varphi_+(x)$

Use Eq. (1) for $K(x-x')$ $(t > t')$ and Eq. (3) for $\varphi_+(t', x')$ in Eq. (4):

$$\phi_+(t,\boldsymbol{x}) = i\int d^3\boldsymbol{x}' \left[\frac{-i}{(2\pi)^3}\int d^3\boldsymbol{p}\,\exp\left[i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}') - iE(t-t')\right]\right.$$
$$\left.\times \left(\frac{\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}{2E}\right)\gamma^0 u(k)\exp[-ik^0t' + i\boldsymbol{k}\cdot\boldsymbol{x}']\right].$$

Collect terms in x' and move terms in x outside the x' integral:

$$\phi_+(t,\boldsymbol{x}) = \int d^3\boldsymbol{p}\left[\frac{1}{(2\pi)^3}\int d^3\boldsymbol{x}'\,\exp(i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{x}')\right]\left(\frac{\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}{2E}\right)$$
$$\times\ \exp(i(E-k^0)t')\gamma^0 u(k)\exp(i\boldsymbol{p}\cdot\boldsymbol{x} - iEt)$$

Term in square brackets is $\delta^3(\boldsymbol{k}-\boldsymbol{p})$

Proof of integral formula for $\varphi_+(x)$ (ii)

Use the delta function to carry out integral over $\boldsymbol{p}$:

$$\phi_+(t, \boldsymbol{x}) = \exp(-ik^0 t + i\boldsymbol{k}\cdot\boldsymbol{x}) \frac{(\gamma^0 k^0 - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)\gamma^0 u(k)}{2k^0}$$

Used $E = \sqrt{|\boldsymbol{p}|^2 + m^2}$ so that when delta function enforced $\boldsymbol{p} = \boldsymbol{k}$, it also caused E to be replaced by k^0.

Now use fact that $u(k)$ is solution of free-particle Dirac eq.

$$(\not{k} - m)u(k) = (\gamma^0 k_0 - \boldsymbol{\gamma}\cdot\boldsymbol{k} - m)u(k) = 0$$

Therefore $(\gamma^0 k^0 - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)\gamma^0 u(k) = \gamma^0(\gamma^0 k_0 + \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)u(k) = 2k_0 u(k)$

(Used $\gamma^i\gamma^0 = -\gamma^0\gamma^i$ and $(\boldsymbol{\gamma}\cdot\boldsymbol{k} + m)u(k) = \gamma^0 k_0 u(k)$)

Therefore $\phi_+(t, \boldsymbol{x}) = u(k)\exp(-ik^0 t + i\boldsymbol{k}\cdot\boldsymbol{x})$, $\quad t > t'$ (5)

Comparing Eq. (5) with Eq. (3) proves Eq. (4).

Proof of integral formula for $\varphi_+(x)$ (iii)

If $t < t'$ we need Eq. (2) for $K(x-x')$

only difference is this minus sign

$$K(x-x') = -\frac{i}{(2\pi)^3}\int d^3\boldsymbol{p}\,\exp\left[i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}') - iE(t-t')\right]\left(\frac{-\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m}{2E}\right), \qquad t < t' . \qquad (2)$$

Calculation similar to before but now

$$(-\gamma^0 k^0 - \boldsymbol{\gamma}\cdot\boldsymbol{k} + m)\gamma^0 u(k) = -\gamma^0(\not{k} - m)u(k) = 0$$

Combining results for $t > t'$ and $t < t'$,

$$i\int d^3\boldsymbol{x}'\,K(x-x')\gamma^0\phi_+(t',\boldsymbol{x}') = \begin{cases} \phi_+(t,\boldsymbol{x}) & t > t' , \\ 0 & t < t' . \end{cases}$$

I.e. for $k_0 > 0$, $\varphi_+(t,\boldsymbol{x})$ is given as an integral over $\mathrm{d}^3\boldsymbol{x}'$ at a given time t' in the past ($t > t'$).

Integral formula for $\varphi_{-}(x)$ and adjoint

For the negative energy solutions use $\phi_{-}(x') = v(k)e^{ik\cdot x'}$

This leads to the same as above but $t > t' \leftrightarrow t < t'$,

$$i\int d^3\boldsymbol{x}'\, K(x-x')\gamma^0\phi_{-}(t',\boldsymbol{x}') = \begin{cases} 0 & t > t'\ , \\ \phi_{-}(t,\boldsymbol{x}) & t < t'\ . \end{cases}$$

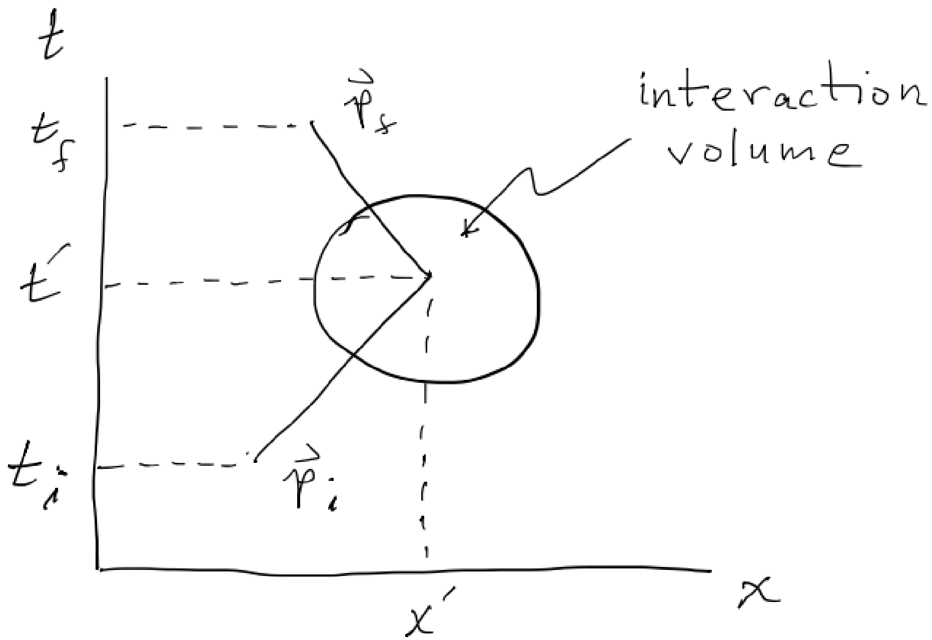
$\longrightarrow$ Positive energy solutions spread out into the future; negative energy solutions spread out into the past.

In similar way, find useful formula for adjoint:

$$i\int d^3\boldsymbol{x}\,\overline{\phi}_{+}(t,\boldsymbol{x})\gamma^0 K(x-x') = \begin{cases} 0 & t' > t\ , \\ \overline{\phi}_{+}(t',\boldsymbol{x}') & t' < t\ . \end{cases}$$

First-order matrix element (Sec. 6.5)

$$S_{fi}^{(1)} = ie \int d^4x' \, \overline{\phi}_f(x') \not{A}(x') \phi_i(x')$$



Derivation of second-order matrix element

Using
$$
\begin{aligned}
S_{fi} &= \int d^3\boldsymbol{x}\, \phi_f^\dagger(x)\phi_i(x) \\
&- e\int d^3\boldsymbol{x}\, \phi_f^\dagger(x) \int d^4x'\, K(x-x')\not{A}(x')\phi_i(x') \\
&+ e^2\int d^3\boldsymbol{x}\, \phi_f^\dagger(x) \int d^4x'\, d^4x'' K(x-x'')\not{A}(x'')K(x''-x')\not{A}(x')\phi_i(x')\ ,
\end{aligned}
$$

2nd order term

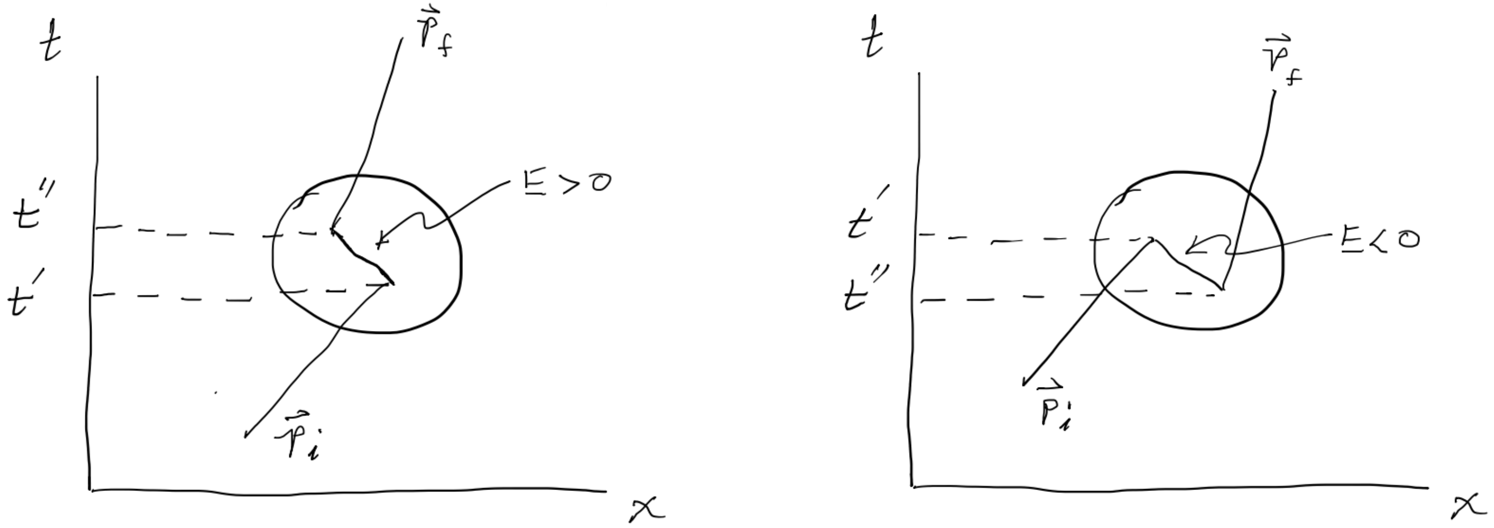
and

$$
\int d^3\boldsymbol{x}\, \phi_f^\dagger(t,\boldsymbol{x})K(x-x') = \int d^3\boldsymbol{x}\, \overline{\phi}_f(t,\boldsymbol{x})\gamma^0 K(x-x') = -i\overline{\phi}_f(t',\boldsymbol{x}')\ , \qquad t' < t\ ,
$$

we find the 2nd-order scattering matrix element:

$$
\begin{aligned}
S_{fi}^{(2)} &= e^2 \int\int d^4x'\, d^4x'' \int d^3\boldsymbol{x}\, \phi_f^\dagger(x)K(x-x'')\not{A}(x'')K(x''-x')\not{A}(x')\phi_i(x') \\
&= -ie^2 \int\int d^4x'\, d^4x'' \overline{\phi}_f(x'')\not{A}(x'')K(x''-x')\not{A}(x')\phi_i(x')\ .
\end{aligned}
$$

Contributions to scattering at 2nd order



Both diagrams are included in $S_{fi}^{(2)}$ since the double integral contains both for $t'' > t'$ and $t'' < t'$.

For $t'' > t'$, $K(x''-x')$ gives propagation of e⁻ with $E>0$ from x' to x''.

For $t'' < t'$, $K(x''-x')$ represents propagation of e⁻ with $E<0$ from x' to x'', or equivalently of e⁺ with $E>0$ from x'' to x'.

If only one time ordering is included, result is not Lorentz invariant.

Feynman rules for QED

- For a given reaction, write down all topologically distinct diagrams for that connect the initial and final state particles using the allowed vertices of the theory. Assign four-momenta to each particle so that it is conserved at each vertex.
- The conventions for lines were shown in Fig. 1.1. For fermions, the arrow points forward in time and for antifermions it points backwards. Photons are drawn with wavey lines.
- For each external fermion line, start at the tip of the arrow and write a Dirac spinor $\overline{u}$ for a particle and $\overline{v}$ for an antiparticle, evaluated with the particle's four-momentum.
- At a fermion-photon vertex, write down the factor $-iq\gamma_\mu$, where q is the charge of the particle, e.g., $ie\gamma_\mu$ for an electron.
- After the vertex factor, if the tail of the fermion line is in the initial or final state, write down a Dirac spinor u for a particle or v for antiparticle.
- For a photon in an initial or final state, include a factor of the polarisation vector ε^μ or $\varepsilon^{\mu *}$, respectively. Its Lorentz index is contracted with that of the gamma matrix at the vertex to which the photon is coupled.
- For an internal photon, include the propagator $-ig^{\mu\nu}/q^2$. The Lorentz indices are contracted with those of the gamma matrices of the vertices at each end of the photon.
- For an internal electron, include the propagator $i(\not{p} + m)/(p^2 - m^2)$.
- For a diagram that differs from another by exchange of an identical fermion, include a minus sign.

 fermion

 antifermion

 photon

incoming (outgoing) fermion : $u(p)$ $\quad (\bar{u}(p))$

" " antifermion : $v(p)$ $\quad (\bar{v}(p))$

" " photon $\varepsilon_\mu(k)$ $\quad (\varepsilon^*_\mu(k))$

↑ polarization vector

virtual photon : $\dfrac{-ig^{\mu\nu}}{q^2 + i\varepsilon}$

virtual electron $\dfrac{i \not{p} + m}{p^2 - m^2 + i\varepsilon}$

fermion – γ vertex : $-i(\pm e)\gamma_\mu (2\pi)^4 \delta^4(p_f - p_i - q)$