PH4442 Advanced Particle Physics 2025/26
Lecture Week 5

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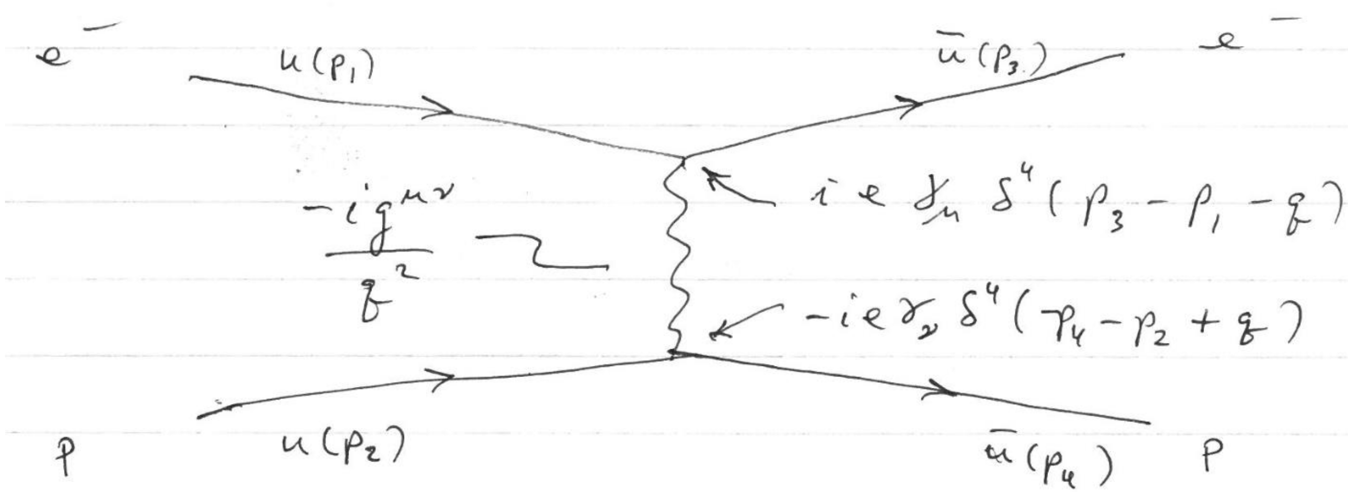
- Summary of Feynman rules
- From amplitudes to observables
- $e^+e^- \to \mu^+\mu^-$
- The ALEPH detector at LEP

Recall S-matrix element for ep $\longrightarrow$ ep

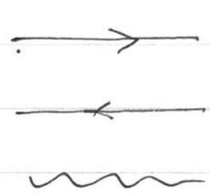
$$S_{fi}^{(1)} = (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)\ [\overline{u}(p_3)(ie\gamma_\mu)u(p_1)]\ \frac{-i}{q^2}\ [\overline{u}(p_4)(-ie\gamma^\mu)u(p_2)]$$



Related to invariant amplitude $\mathcal{M}$ by $\quad S_{fi} = (2\pi)^4 \delta^4(P_f - P_i)\mathcal{M}$

$$\longrightarrow \qquad \mathcal{M} = \frac{-ie^2}{(p_3 - p_1)^2}\,[\overline{u}(p_3)\gamma_\mu u(p_1)]\ [\overline{u}(p_4)\gamma^\mu u(p_2)]$$

Feynman rules



fermion

antifermion

photon

Incoming (outgoing) fermion $u(p)$ $\left(\bar{u}(p)\right)$

" " antifermion $v(p)$ $\left(\bar{v}(p)\right)$

" " photon $\varepsilon_\mu(k)$ $\left(\varepsilon^*_\mu(k)\right)$ ← polarisation vector

(intermediate) virtual photon $\dfrac{-ig^{\mu\nu}}{q^2 + i\varepsilon}$

virtual electron $\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$

} For this course, won't need ε

fermion – γ vertex: $-i(\underbrace{\pm e}_{q})\gamma_\mu$

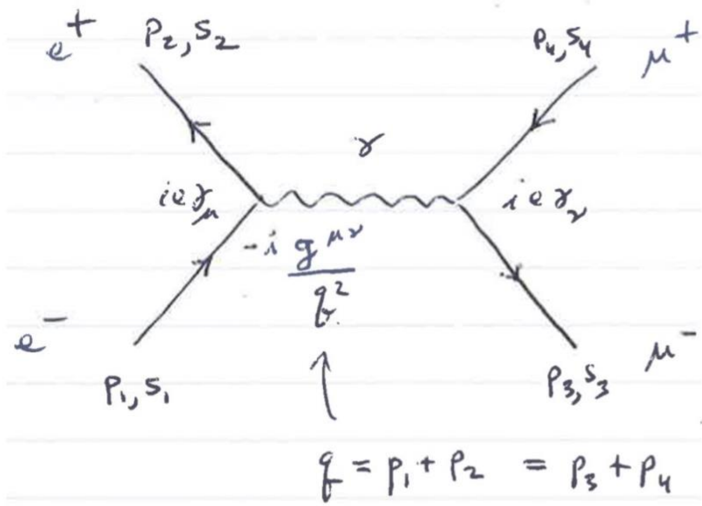
Feynman rules

- For a given reaction, write down all topologically distinct diagrams for that connect the initial and final state particles using the allowed vertices of the theory. Assign four-momenta to each particle so that it is conserved at each vertex.
- The conventions for lines were shown in Fig. 1.1. For fermions, the arrow points forward in time and for antifermions it points backwards. Photons are drawn with wavey lines.
- For each external fermion line, start at the tip of the arrow and write a Dirac spinor $\overline{u}$ for a particle and $\overline{v}$ for an antiparticle, evaluated with the particle's four-momentum.
- At a fermion-photon vertex, write down the factor $-iq\gamma_\mu$, where q is the charge of the particle, e.g., $ie\gamma_\mu$ for an electron.
- After the vertex factor, if the tail of the fermion line is in the initial or final state, write down a Dirac spinor u for a particle or v for antiparticle.
- For a photon in an initial or final state, include a factor of the polarisation vector ε^μ or $\varepsilon^{\mu *}$, respectively. Its Lorentz index is contracted with that of the gamma matrix at the vertex to which the photon is coupled.
- For an internal photon, include the propagator $-ig^{\mu\nu}/q^2$. The Lorentz indices are contracted with those of the gamma matrices of the vertices at each end of the photon.
- For an internal electron, include the propagator $i(\not{p} + m)/(p^2 - m^2)$.
- For a diagram that differs from another by exchange of an identical fermion, include a minus sign.

Amplitude for $e^+e^- \rightarrow \mu^+\mu^-$

In QED only one leading-order diagram (SM also has annihilation into Z).

Assign four-momenta, enforce conservation at every vertex.



Follow the fermion lines backwards, bar at head, no bar at tail, u for particle, v for antiparticle:

$$\mathcal{M} = [\overline{v}(p_2, s_2)(ie\gamma_\mu)u(p_1, s_1)] \, \frac{-ig^{\mu\nu}}{q^2} \, [\overline{u}(p_3, s_3)(ie\gamma_\nu)v(p_4, s_4)]$$

Contract Lorentz index ν, use $q^2 = E_{\rm cm}{}^2 \equiv s$:

$$\mathcal{M} = i\frac{e^2}{s} \, [\overline{v}(p_2, s_2)\gamma_\mu u(p_1, s_1)] \, [\overline{u}(p_3, s_3)\gamma^\mu v(p_4, s_4)]$$

Trace theorems

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu) &= \mathrm{Tr}(\text{any product of odd number of } \gamma \text{ matrices}) = 0\,, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}\,, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\right]\,, \\
\mathrm{Tr}(\gamma^5) &= 0\,, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}\,,
\end{aligned}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon^{0123} = 1$.

Useful variants:

$$\begin{aligned}
\mathrm{Tr}(\not{p}\gamma^\mu\not{k}\gamma^\nu) &= 4\left[p^\mu k^\nu + p^\nu k^\mu - (p\cdot k)g^{\mu\nu}\right]\,, \\
\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) &= 4\left[(a\cdot b)(c\cdot d) + (a\cdot d)(b\cdot c) - (a\cdot c)(b\cdot d)\right]
\end{aligned}$$

Spin averaged $|\mathcal{M}|^2$ for $e^+e^- \to \mu^+\mu^-$

Square the amplitude:

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\,[\overline{v}_2\gamma_\mu u_1][\overline{v}_2\gamma_\nu u_1]^*\;[\overline{u}_3\gamma^\mu v_4][\overline{u}_3\gamma^\nu v_4]^*$$

Use Casimir's trick (neglect rest masses)

$$\sum_{s_1,s_2}[\overline{u}_1\Gamma_A u_2][\overline{u}_1\Gamma_B u_2]^* \approx \mathrm{Tr}[\not{p}_1\Gamma_A\not{p}_2\overline{\Gamma}_B]$$

to average over initial-state, sum over final-state spins:

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\mathrm{Tr}[\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu]\,\mathrm{Tr}[\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu]$$

Spin averaged $|\mathcal{M}|^2$

Use trace theorems to evaluate:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{4e^4}{s^2}[p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot p_2)][p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - g^{\mu\nu}(p_3 \cdot p_4)] \\ &= \frac{4e^4}{s^2}[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) + (g_{\mu\nu}g^{\mu\nu} - 4)(p_1 \cdot p_2)(p_3 \cdot p_4)]\end{aligned}$$

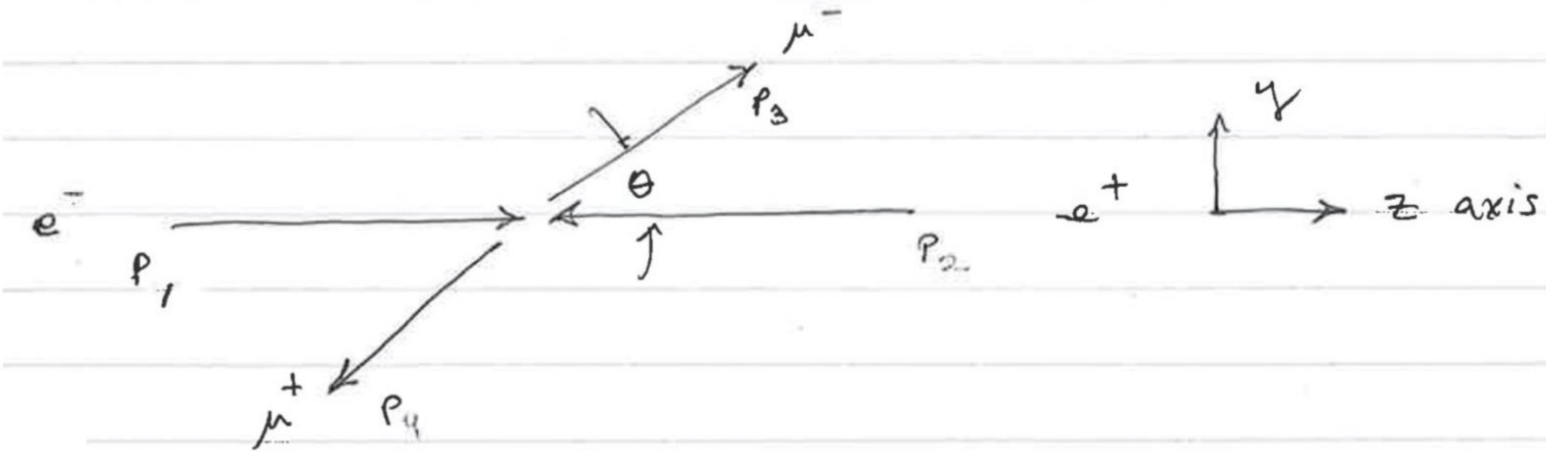
$$g_{\mu\nu}g^{\mu\nu} - 4 = 0$$

Use $\alpha = e^2/4\pi = 1/137$,

$$\langle|\mathcal{M}|^2\rangle = \frac{128\pi^2\alpha^2}{s^2}[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

Go to c.m. frame

Consider the c.m. frame:



Let $E = E_{\rm beam} = E_{\rm cm}/2$:

$$p_1 = (E, 0, 0, E)$$

$$p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, 0, E\sin\theta, E\cos\theta)$$

(suppose $\mu^+\mu^-$ in (y,z) plane)

$$p_4 = (E, 0, -E\sin\theta, -E\cos\theta)$$

Differential cross section for $e^+e^- \to \mu^+\mu^-$

The required four-vector products are:

$$p_1 \cdot p_3 = p_2 \cdot p_4 = E^2(1 + \cos\theta) ,$$
$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2(1 - \cos\theta) ,$$
$$p_1 \cdot p_2 = p_3 \cdot p_4 = 2E^2 .$$

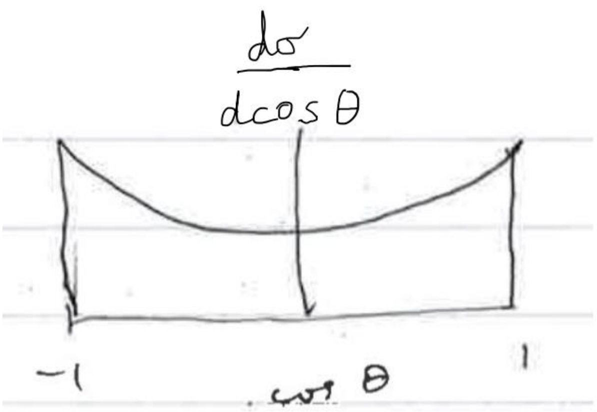
giving $\langle|\mathcal{M}|^2\rangle = 16\pi^2\alpha^2(1 + \cos^2\theta)$

Use in the equation for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 E_{\rm cm}^2} = \frac{\alpha^2}{4E_{\rm cm}^2}(1 + \cos^2\theta)$$

Integrate over φ:

$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2\pi}{2E_{\rm cm}^2}(1 + \cos^2\theta)$$



Total cross section for $e^+e^- \rightarrow \mu^+\mu^-$

Integrate differential cross section over $\cos\theta$:

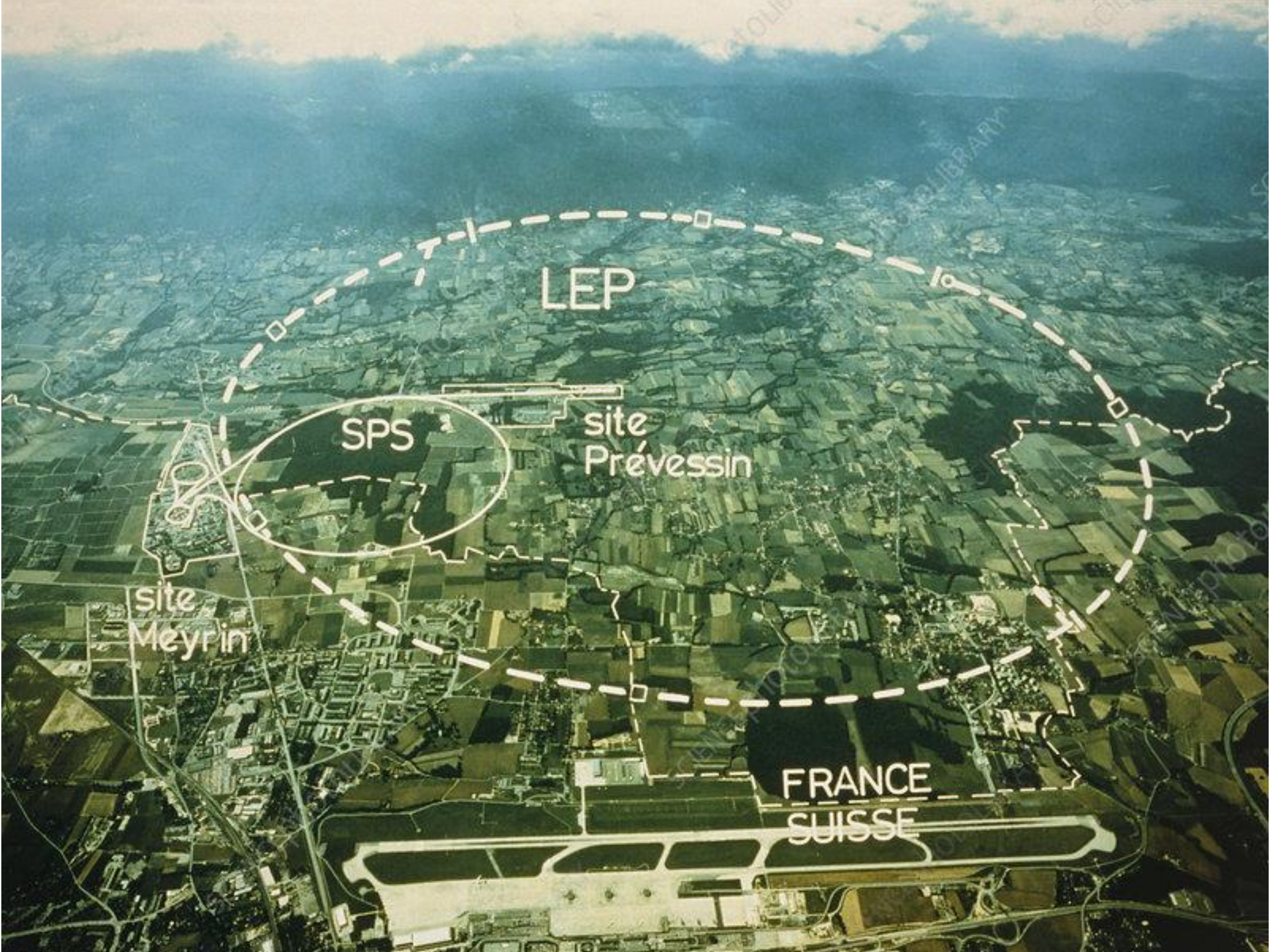
$$\sigma = \int_{-1}^{1} \frac{d\sigma}{d\cos\theta}\, d\cos\theta = \frac{\pi\alpha^2}{2E_{\rm cm}^2} \int_{-1}^{1} (1+x^2)\, dx = \frac{4\pi\alpha^2}{3E_{\rm cm}^2}$$

In experimentalists units (energy $^{-2} \rightarrow$ area): $\sigma = \frac{4\pi\alpha^2\hbar^2 c^2}{3E_{\rm cm}^2}$

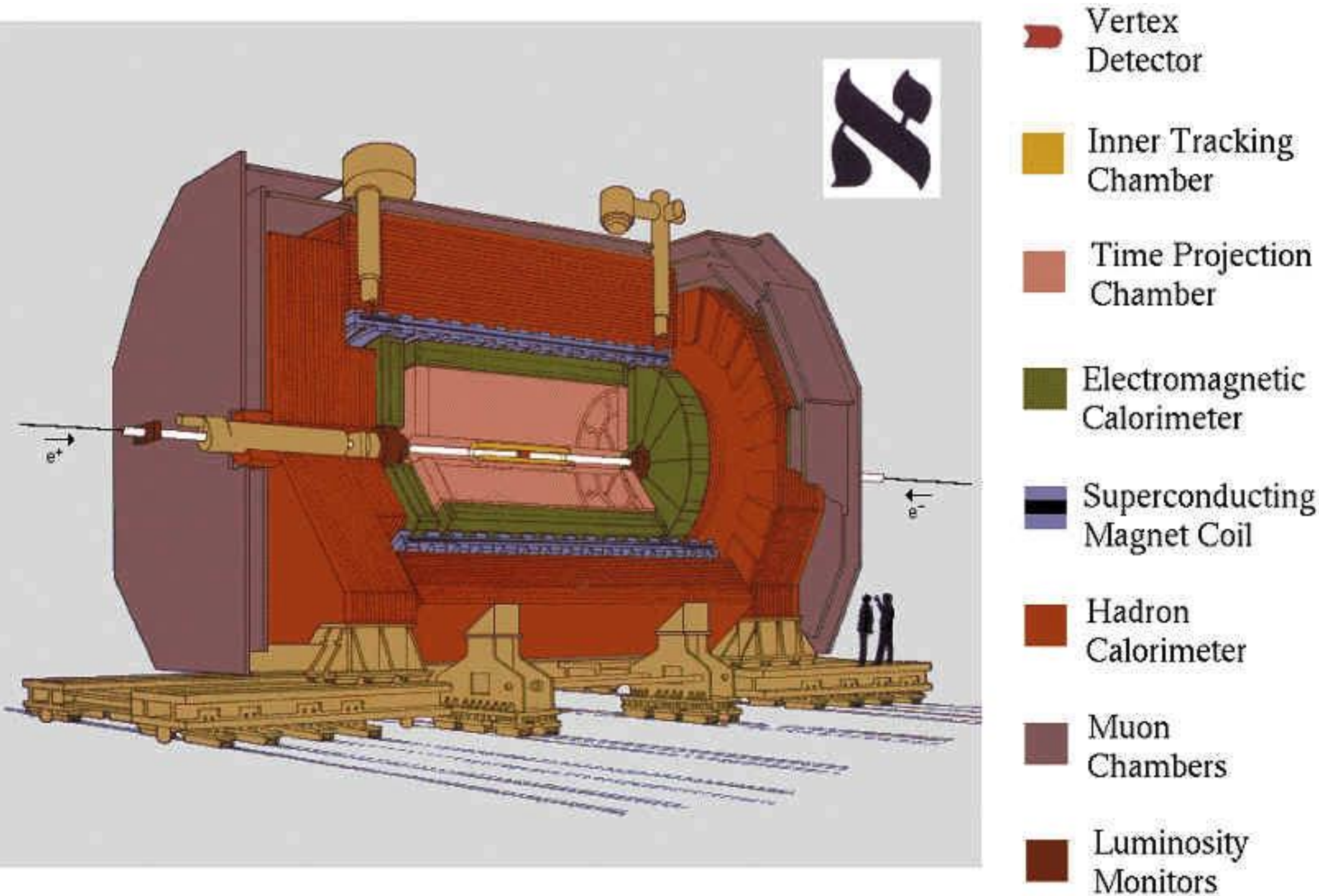
TPC/2γ Experiment at SLAC (1984-86) collected data at $E_{\rm cm}$ = 29 GeV with integrated luminosity $\int L\, dt = 60\ \text{pb}^{-1}$

$$\sigma = 0.103\,\text{nb}$$

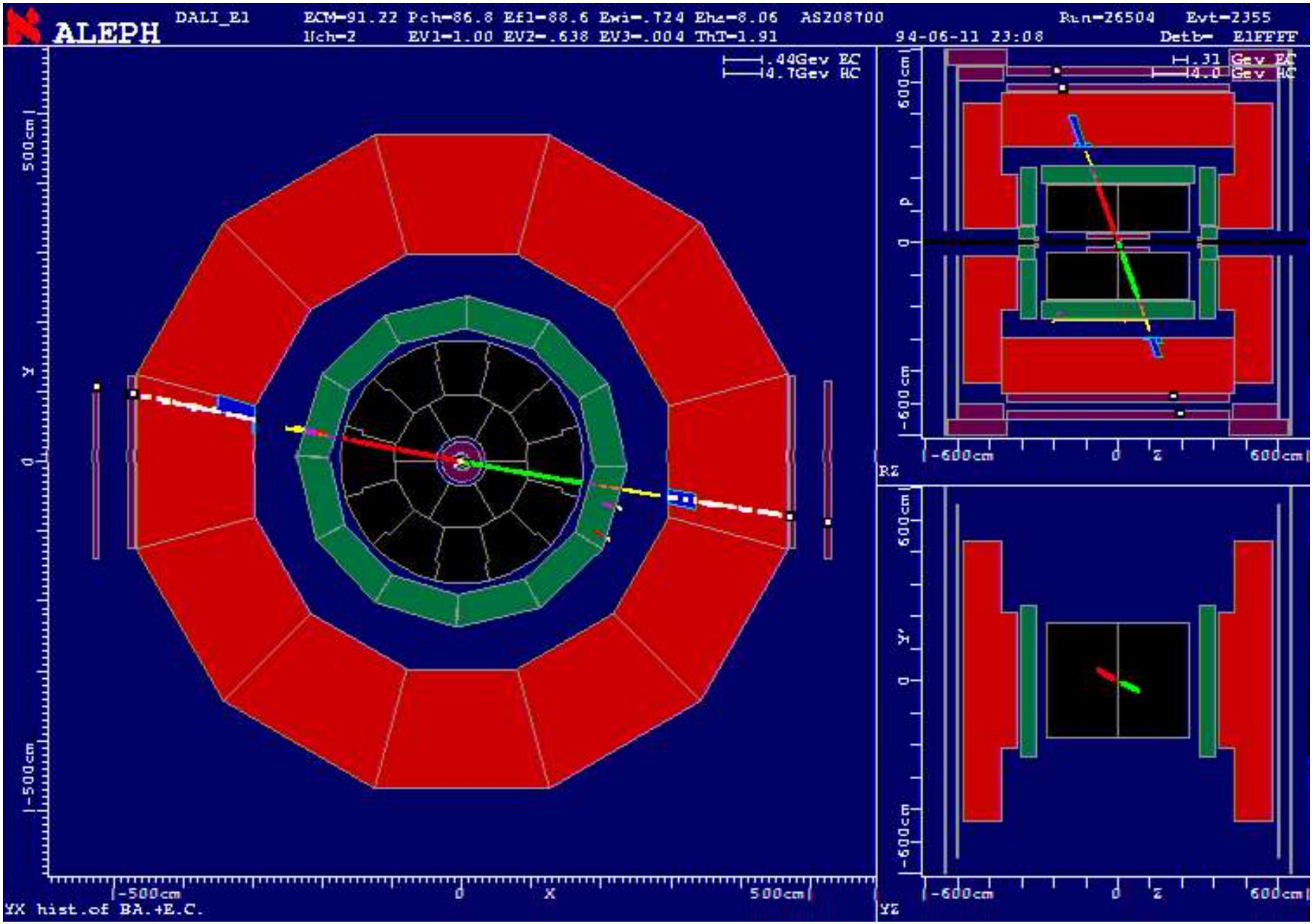
$$N = \sigma \int \mathcal{L}\, dt = 0.103\,\text{nb} \;\times\; 60\,\text{pb}^{-1} \times \frac{1000\,\text{nb}^{-1}}{1\,\text{pb}^{-1}} = 6\,180\ \text{events}$$


LEP
SPS
site
Prévessin
site
Meyrin
FRANCE
SUISSE

The ALEPH detector at LEP



$e^+e^- \rightarrow \mu^+\mu^-$ in the ALEPH detector at LEP



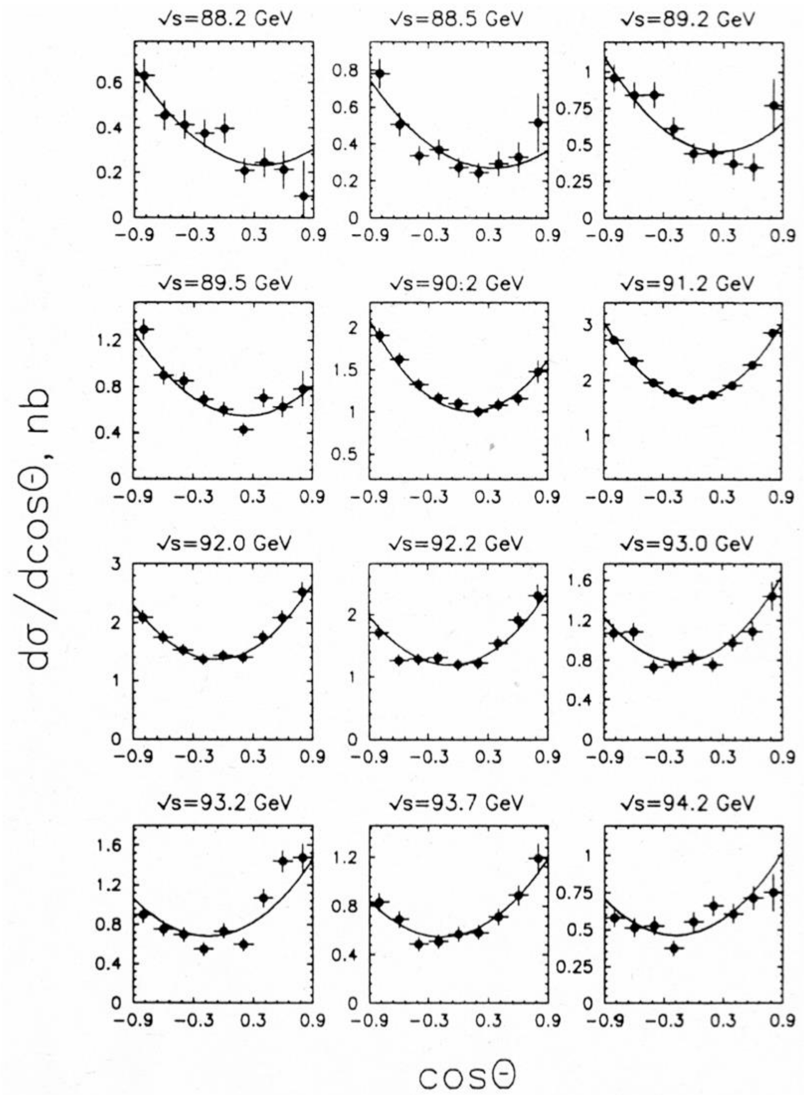


Figure 9.14: Differential cross sections $d\sigma/d\cos\theta$ for $e^+e^- \to l^+l^-$ for all charged leptons combined (from [80]).

$e^+e^- \to \mu^+\mu^-$ in the JADE detector at PETRA

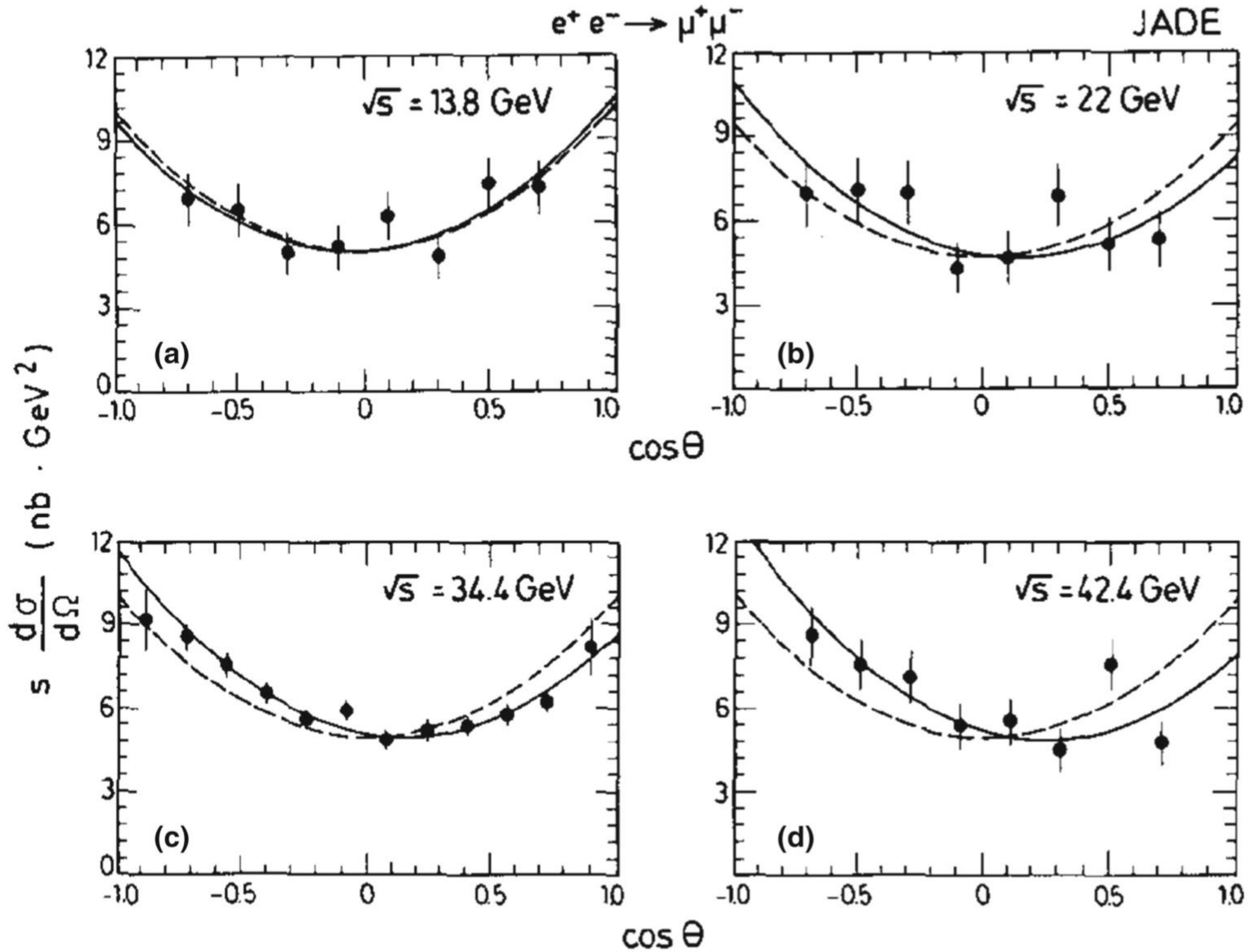


Fig. 19 Angular distributions of $e^+e^- \to \mu^+\mu^-$ for four c.m. energies. The dashed lines are symmetric fits $f(\theta) \propto (1 + \cos^2\theta)$, the full lines are fits allowing an additional asymmetry $f(\theta) \propto (1 + \cos^2\theta + B\cos\theta)$ [95]