Solving the Dirac Equation

1 Introduction

The goal is to find the solutions to the Dirac equation for a free particle of mass $m$,

$$(i\slashed{\partial} - m)\psi(x) = 0,$$  \hspace{1cm} (1)

where the slash notation is $\slashed{\partial} = \gamma^\mu \partial_\mu$, $\gamma^\mu$ are the $4 \times 4$ Dirac gamma matrices and $\psi(x)$ is a four-component Dirac spinor.

2 Components of Dirac equation as solutions to the Klein-Gordon equation

As a first step, we will show that all components of $\psi$ individually satisfy the Klein-Gordon equation. To do this we apply $(i\slashed{\partial} + m)$ to Eq. (1),

$$(i\slashed{\partial} + m)(i\slashed{\partial} - m)\psi(x) = 0,$$  \hspace{1cm} (2)

which gives

$$(\slashed{\partial}^2 + m^2)\psi(x) = (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi(x) = 0.$$  \hspace{1cm} (3)

The derivatives $\partial_\mu \partial_\nu$ are symmetric under interchange of $\mu$ and $\nu$, and this is contracted with $\gamma^\mu \gamma^\nu$, which can written as a sum of symmetric and antisymmetric parts as

$$\gamma^\mu \gamma^\nu = \frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) + \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2} [\gamma^\mu, \gamma^\nu].$$  \hspace{1cm} (4)

The commutator is antisymmetric in $\mu$ and $\nu$, and so contracting with the symmetric $\partial_\mu \partial_\nu$ gives zero. For the anticommutator we can use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$. We therefore find

$$\left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) + m^2\right] \psi = (g^{\mu\nu} \partial_\mu \partial_\nu + m^2)\psi = (\partial_\mu \partial_\mu + m^2)\psi = 0.$$  \hspace{1cm} (5)

This actually represents four equations, i.e., a $4 \times 4$ matrix multiplying the four-component spinor $\psi$. Thus each component of $\psi$ individually satisfies the Klein-Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\psi_i = 0, \quad i = 1, \ldots, 4.$$  \hspace{1cm} (6)
3 Dirac spinors $u$ and $v$

Since each component of the solution to the Dirac equation satisfies the Klein-Gordon equation (6), we will have plane-wave solutions for $\psi(x)$ of the form

$$\psi_+(x) = u(p)e^{-ip\cdot x}, \quad (7)$$

$$\psi_-(x) = v(p)e^{ip\cdot x}. \quad (8)$$

The solutions $\psi_+$ and $\psi_-$ have energies $E = p^0$ and $E = -p^0$, respectively. Here the energy $E$ means the eigenvalue of $i\partial/\partial t$ and can be positive or negative, while $p^0 \equiv +\sqrt{p^2 + m^2} > 0$. That is, $\psi_+$ has the “normal” time dependence for a positive energy particle, $\psi_+ \sim e^{-ip^0t}$, while $\psi_- \sim e^{+ip^0t}$ corresponds to a negative energy. This will be re-interpreted later to describe antiparticles.

The solutions $\psi_+$ and $\psi_-$ thus suppose that the time and space dependence is given by the exponential factor $e^{\mp ip\cdot x}$ and that this multiplies a four-component spinor, $u(p)$ or $v(p)$, that depends only on the particle’s four-momentum. By substituting $\psi_+$ into the Dirac equation we find

$$(i\partial - m)u(p)e^{-ip\cdot x} = 0, \quad (9)$$

which gives

$$(i(-i\partial) - m)u(p)e^{-ip\cdot x} = 0, \quad (10)$$

The exponential term can be canceled and the analogous procedure carried out for $\psi_-$, leading to the equations for $u(p)$ and $v(p)$,

$$(\not{p} - m)u(p) = 0, \quad (11)$$

$$(\not{p} + m)v(p) = 0. \quad (12)$$

In a similar way one finds for the Dirac adjoint spinors $\overline{u} = u^\dagger\gamma^0$ and $\overline{v} = v^\dagger\gamma^0$

$$\overline{u}(\not{p} - m) = 0, \quad (13)$$

$$\overline{v}(\not{p} + m) = 0. \quad (14)$$

4 Rest-frame solutions

First consider the rest frame, where the particle’s four-momentum is $p^\mu = (p^0, 0, 0, 0) = (m, \mathbf{0})$. In this frame we want to solve Eqs. (11) and (12). Using $\not{p} = \gamma^\mu p_\mu = m\gamma^0$ and canceling a factor of $m$ we obtain
\[(\gamma^0 - I)u(m, 0) = 0, \quad (15)\]
\[(\gamma^0 + I)v(m, 0) = 0. \quad (16)\]

In the Dirac representation for the gamma matrices, \(\gamma^0\) is

\[
\gamma^0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad (17)
\]

so that equation (15) becomes

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -2
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = 0, \quad (18)
\]

where \(a, b, c\) and \(d\) are the four components of \(u\) that we want to find. Equation (18) is solved for arbitrary \(a\) and \(b\) with \(c = d = 0\). We thus can choose two linearly independent solutions to be

\[
u_s(m, 0) = \begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = N \begin{pmatrix}
\varphi_s \\
0
\end{pmatrix}, \quad (19)
\]

where \(N\) is a normalization constant and the index \(s = 1, 2\) labels the two independent two-component Weyl spinors. These are chosen to be orthogonal and normalized to unity, i.e., \(\varphi^\dagger_r \varphi_s = \delta_{rs}\). We can take \(\varphi_1\) and \(\varphi_2\) to be, for example,

\[
\varphi_1 = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad \varphi_2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}. \quad (20)
\]

In a similar way one can find the rest-frame solution for the spinor \(v_s\) to be

\[
v_s(m, 0) = N \begin{pmatrix}
0 \\
\chi_s
\end{pmatrix}, \quad (21)
\]

where the two-component spinors \(\chi_s\) can be taken as

\[
\chi_1 = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad \chi_2 = \begin{pmatrix}
1 \\
0
\end{pmatrix}. \quad (22)
\]

For \(\chi_s\) the order of the 1 and 0 in the upper and lower elements is reversed compared to \(\varphi_s\); this is so that we can identify later the solutions \(u_1\) and \(v_1\) as representing particles with spin parallel to the \(z\) axis. Note that fixing the normalization of the Weyl spinors such that \(\varphi^\dagger_r \varphi_s = \chi^\dagger_r \chi_s = \delta_{rs}\) does not fix the normalization of \(u_s\) or \(v_s\).
5 Solutions for four-momentum $p$

Before dealing with the normalization, we can first work out the solutions in a reference frame where the particle has four-momentum $p$. That is, we want to find the solutions $u_s(p)$ and $v_s(p)$ to Eqs. (11) and (12). These can be derived from the rest-frame solutions as

$$u_s(p) = (\not{p} + m)u_s(m,0), \quad (23)$$
$$v_s(p) = (\not{p} - m)v_s(m,0). \quad (24)$$

To see that this actually gives the correction solutions, we can apply $(\not{p} - m)$ to the left-hand side of Eq. (23):

$$(\not{p} - m)u_s(p) = (\not{p} - m)(\not{p} + m)u_s(m,0)$$
$$= (\not{p}\not{p} - m^2)u_s(m,0)$$
$$= (p^2 - m^2)u_s(m,0)$$
$$= 0. \quad (28)$$

Here we used $\not{p}\not{p} = p_\mu p^\mu = p^2$, which can be easily verified by using the anticommutation properties of the gamma matrices. The final equality above follows from the fact that $p^2 = m^2$ for a free on-shell particle. The analogous result follows for $v_s(p)$ and thus Eqs. (23) and (24) allow one to obtain the solutions for a particle with four-momentum $p$ directly from the rest-frame solutions.

To write down the solutions explicitly we can use the Dirac representation of the gamma matrices, where

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3, \quad (29)$$

and therefore $\not{p} = \gamma^\mu p_\mu$ is

$$\not{p} = \begin{pmatrix} p^0 & -\sigma \cdot p \\ \sigma \cdot p & -p^0 \end{pmatrix}. \quad (30)$$

The unnormalized solutions are thus found from Eqs. (23) and (24) to be

$$u_s(p) = \begin{pmatrix} p^0 + m & -\sigma \cdot p \\ \sigma \cdot p & -p^0 + m \end{pmatrix} \begin{pmatrix} \varphi_s \\ 0 \end{pmatrix} = \begin{pmatrix} (p^0 + m)\varphi_s \\ (\sigma \cdot p)\varphi_s \end{pmatrix}, \quad (31)$$
$$v_s(p) = \begin{pmatrix} p^0 - m & -\sigma \cdot p \\ \sigma \cdot p & -p^0 - m \end{pmatrix} \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} = \begin{pmatrix} (\sigma \cdot p)\chi_s \\ (p^0 + m)\chi_s \end{pmatrix}. \quad (32)$$
6 Normalization of spinors

To choose the normalization of the Dirac spinors we recall that $\rho = \psi^\dagger \psi$ will be interpreted as a probability density, and furthermore this density is the zeroth component of the four-vector current $j^\mu = (\rho, j)$. We need the probability to find the particle inside a specified volume to be Lorentz invariant. This probability is

$$P = \int d^3x \, j^0(x) = \int d^3x \, \psi^\dagger(x)\psi(x) = \begin{cases} \int d^3x \, u^\dagger u & \text{for } \psi_+ \\ \int d^3x \, v^\dagger v & \text{for } \psi_- \end{cases}. \quad (33)$$

One can show (see, e.g., Ref. [1]) that this will be Lorentz invariant if $\psi^\dagger \psi$ transforms like the zeroth component of a four-vector. And the only four-vector on which this can depend is $p = (p^0, \vec{p})$, i.e., the four-momentum of the particle. We therefore take

$$u^\dagger u \propto v^\dagger v \propto p^0 \equiv E. \quad (34)$$

The normalization condition also takes on a simple form when expressed in terms of the Dirac adjoint spinors $\overline{u} = u^\dagger \gamma^0$ and $\overline{v} = v^\dagger \gamma^0$. One can easily show that (34) implies

$$\overline{u}_s u_s = \overline{v}_s v_s = 2E \delta_{rs}, \quad (35)$$

$$\overline{v}_s v_s = -2m \delta_{rs}. \quad (36)$$

Using this normalization and the Pauli matrices for the terms involving $\vec{\sigma} \cdot \vec{p}$ allows us to write the complete solutions for the spinors $u$ and $v$ as

$$u_1(p) = \sqrt{E + m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E + m} \\ \frac{p_x + ip_y}{E + m} \end{pmatrix}, \quad u_2(p) = \sqrt{E + m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E + m} \\ -\frac{p_z}{E + m} \end{pmatrix}, \quad (37)$$

$$v_1(p) = \sqrt{E + m} \begin{pmatrix} \frac{p_x - ip_y}{E + m} \\ \frac{-p_z}{E + m} \\ 0 \\ 1 \end{pmatrix}, \quad v_2(p) = \sqrt{E + m} \begin{pmatrix} \frac{p_x + ip_y}{E + m} \\ \frac{p_z}{E + m} \\ 0 \\ 1 \end{pmatrix}. \quad (38)$$

Here $E = +\sqrt{\vec{p}^2 + m^2}$ is positive. The full solutions to the Dirac equation are obtained by combining the spinors $u_s$ or $v_s$ with the exponential term $e^{-ip \cdot x}$ or $e^{ip \cdot x}$, respectively, according to Eqs. (7) and (8). The solutions also implicitly contain a factor $1/\sqrt{V}$ to be normalized in a volume $V$. For now it is convenient to drop this factor, i.e., we take $V = 1$.

References