PH4442 Advanced Particle Physics
Problem sheet 1
Due Monday, 20 October 2025

For all exercises take $\hbar = c = 1$ and assume the Einstein summation convention over repeated upper and lower Lorentz indices, e.g., $p_\mu p^\mu = \sum_{\mu=0}^{3} p_\mu p^\mu$.

Exercise 1 [4 marks] Consider the Lorentz transformation corresponding to a boost along the z axis:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

By defining the rapidity $\omega = \cosh^{-1}\gamma$, show that the transformation for $x^\mu = (t, x, y, z)$ can be written $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with

$$\Lambda = \begin{pmatrix} \cosh\omega & 0 & 0 & -\sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} .$$

Exercise 2 [10 marks] Suppose the function $\varphi(x)$ is a Lorentz scalar, i.e., under a Lorentz transformation $x' = \Lambda x$, one has $\varphi'(x') = \varphi(x)$. Show that $(\partial_\mu \partial^\mu)\varphi(x)$ transforms as a Lorentz scalar. (It may be easier to start with $(\partial_\mu \partial^\mu)\varphi(x)$ and show this transforms into $(\partial'_\mu \partial'^\mu)\varphi'(x')$.)

Exercise 3(a) [6 marks]: Consider the solution φ to the Klein-Gordon equation for a free particle,

$$(\partial_\mu \partial^\mu + m^2)\varphi = 0 ,$$

as well as the complex-conjugate solution φ^*. The four-vector probability current $j^\mu = (\rho, \vec{j})$ for the Klein-Gordon equation is

$$j^\mu = i\left(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*\right) .$$

In analogy with the continuity relation for probability found for the Schrödinger equation, show that $\partial_\mu j^\mu = \partial\rho/\partial t + \nabla \cdot \vec{j} = 0$.

3(b) [2 marks] We can take the free-particle solutions to be $\varphi(x) = N e^{-ip\cdot x}$ where $x = (t, \vec{x})$ and $p = (E, \vec{p})$ are the usual position and momentum four vectors and N is a normalisation factor. For this solution, find the four-vector probability current j^μ as a function of p^μ. In what important way is the result problematic? (See the lecture notes from week 1.)

Exercise 4 [10 marks] Recall the 4×4 Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \quad i = 1, 2, 3 ,$$

where I_2 represents a 2×2 identity matrix and σ_i are the Pauli matrices. Show that the γ matrices satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, $g = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor of special relativity. Use the known properties of the Pauli matrices, e.g., $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_2$.

Exercise 5 [8 marks] Using the formulas for the Dirac spinors u_1 and u_2 from the lectures, show that $u_1^\dagger u_1 = 2E$ and $u_1^\dagger u_2 = 0$. These are special cases of

$$u_r^\dagger(p) u_s(p) = v_r^\dagger(p) v_s(p) = 2E\delta_{rs} \ .$$

I.e., show explicitly that the relation holds for (a) u_1 with u_1 and (b) u_1 with u_2.

Exercise 6: Consider a plane-wave solution to the Dirac equation $\psi(x) = u(p)e^{-ip\cdot x}$. We have seen that the current $j^\mu = \overline{\psi}\gamma^\mu\psi = \overline{u}\gamma^\mu u$ transforms like a four-vector, and therefore we must have

$$\overline{u}\gamma^\mu u = Ap^\mu$$

for some Lorentz scalar A, since the four-momentum p^μ is the only four-vector on which this could possibly depend.

6(a) [4 marks] Using $u^\dagger u = 2E$, show that $A = 2$ and thus $j^\mu = 2p^\mu$.

6(b) [6 marks] By contracting the equation above with p_μ, show that $\overline{u}u = (m/E)u^\dagger u = 2m$. (In a similar way one finds $\overline{v}v = -(m/E)v^\dagger v = -2m$.)

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