PH4442 Advanced Particle Physics
Problem sheet 2
Due Monday, 3 November 2025

Exercise 1 [6 marks]: Show that the adjoints of the Dirac gamma matrices are ${\gamma^0}^\dagger = \gamma^0$ and ${\gamma^i}^\dagger = -\gamma^i$, $i = 1, 2, 3$. Thus show that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. (For example, write the gamma matrices in terms of the Pauli matrices and use the fact that $\sigma_i^\dagger = \sigma_i$, $i = 1, 2, 3$.)

2(a) **[4 marks]** Consider the positive energy solution to the Dirac equation $\psi_+ = u_1(p)e^{-ip\cdot x}$ for a particle moving with momentum p_z and $p_x = p_y = 0$. Using the formula for $u_1(p)$ and the operator

$$S_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} ,$$

find the component of spin along the z axis.

2(b) **[4 marks]** Consider the negative energy solution $\psi_- = v_1(p)e^{ip\cdot x}$ again for a particle moving with momentum p_z and $p_x = p_y = 0$. Using the formula for $v_1(p)$ find the component of spin along the z axis. (As we interpret ψ_- as a negative energy particle running backwards in time, the component of spin relative to the z axis for the corresponding positron is the opposite of this.)

3(a) **[6 marks]** Consider the Lorentz transformation for a Dirac spinor given by the matrix

$$S = \exp\left(-\frac{\omega}{2}\gamma^0\gamma^2\right) .$$

Show that this corresponds to a Lorentz boost along the y direction with rapidity ω, and find the corresponding Lorentz transformation matrix $\Lambda^\mu{}_\nu$.

3(b) **[4 marks]** Starting from the spinor for a particle at rest $u_1(0)$, boost to a frame with rapidity $-\omega$, so in the new frame the particle itself has rapidity ω. Find the resulting Dirac spinor $u_1(p)$ and show that this agrees for the solutions to the Dirac equation found in the lectures (for $p_x = p_z = 0$). (Use the formulas from the lectures to relate ω, E and p_y.)

Exercise 4: For the calculations below it is sufficient to use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$ and other properties we have derived without writing out the gamma matrices explicitly.

4(a) [**2 marks**] Find $\gamma^\mu \gamma_\mu$.

4(b) [**4 marks**] Find $\gamma^\mu \gamma^\nu \gamma_\mu$.

4(c) [**6 marks**] For $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ show that $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^2 = I$ and $\gamma^{5\dagger} = \gamma^5$.

Exercise 5: The wave equation $W^\nu(x)$ for a vector boson of mass M that couples to a current J^μ has the form

$$\left[(\partial_\alpha \partial^\alpha + M^2) g^{\mu\nu} - \partial^\mu \partial^\nu\right] W_\nu(x) = J^\mu(x) \ ,$$

In analogy with the calculation done in the lectures to find the photon propagator, we will find the corresponding propagator for the massive vector boson.

5(a) [**4 marks**] Start by considering the differential equation for the Green function

$$\left[(\partial_\alpha \partial^\alpha + M^2) g^{\mu\nu} - \partial^\mu \partial^\nu\right] D_{\nu\lambda}(x - x') = \delta^\mu{}_\lambda \delta^4(x - x') \ .$$

Verify that the solution $W^\nu(x)$ satisfies

$$W_\nu(x) = \int d^4x' \, D_{\nu\lambda}(x - x') J^\lambda(x');.$$

5(b) [**4 marks**] Express $D_{\nu\lambda}$ in terms of its Fourier transform $\tilde{D}_{\nu\lambda}(q)$ and show that it satisfies

$$\left[(-q^2 + M^2) g^{\mu\nu} + q^\mu q^\nu\right] \tilde{D}_{\nu\lambda}(q) = \delta^\mu{}_\lambda \ .$$

5(c) [**6 marks**] Suppose $\tilde{D}_{\nu\lambda} = A g_{\nu\lambda} + B q_\nu q_\lambda$ for some values of A and B that can only depend on Lorentz scalars. By substituting this into the equation above, show that the propagator is

$$\tilde{D}_{\nu\lambda} = \frac{-g_{\nu\lambda} + \frac{q_\nu q_\lambda}{M^2}}{q^2 - M^2 + i\varepsilon}$$

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