PH4442 Advanced Particle Physics
Problem sheet 3
Due Monday, 17 November 2025

For the following exercises, consider the scattering reaction $e^- p \to e^- p$, where we model the proton as a point-like Dirac fermion. Label the four-momenta and spins:

incoming e^-	p_1, s_1
incoming proton	p_2, s_2
outgoing e^-	p_3, s_3
outgoing proton	p_4, s_4

First we will first obtain a Lorentz invariant expression for the differential cross section using *Mandelstam variables* s, t and u, defined as

$$
\begin{aligned}
s &= (p_1 + p_2)^2 \;, &(1)\\
t &= (p_1 - p_3)^2 \;, &(2)\\
u &= (p_1 - p_4)^2 \;. &(3)
\end{aligned}
$$

Here $s = E_{\rm cm}^2$ and t can be related to the scattering angle of the outgoing electron. Following along the same lines as seen for the differential cross section $d\sigma/d\Omega$ in the c.m. frame, one can show that $d\sigma/dt$ is related to the invariant amplitude $\mathcal{M}$ by

$$
\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda(s, m_1^2, m_2^2)} |\mathcal{M}|^2 \;,
$$

where $\lambda(s, m_1^2, m_2^2) = (s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2$ is the Källén triangle function.

We will then consider the reaction in the lab frame where the proton is initially at rest. We will neglect the mass of the electron but not the proton mass M. The four-momenta of the incoming and outgoing electron and proton are thus

$$
\begin{aligned}
p_1 &= (E_1, 0, 0, E_1) &(4)\\
p_2 &= (M, 0, 0, 0) &(5)\\
p_3 &= (E_3, 0, E_3 \sin\theta, E_3 \cos\theta) &(6)\\
p_4 &= (E_4, 0, p_{4,y}, p_{4,z}) &(7)
\end{aligned}
$$

where θ is the electron's scattering angle relative to the z axis.

Exercise 1 [4 marks]: Write down the lowest-order Feynman diagram for the reaction, labeling the particles, their momenta and spins. Write down the invariant amplitude $\mathcal{M}$, and indicate what terms correspond to what parts of the diagram.

Exercise 2 [10 marks]: Using the Casimir trick and trace theorems from the lectures, show that the spin-averaged invariant amplitude squared can be written $\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{1}{4} e^4/((q^2)^2) L_{\mu\nu} H^{\mu\nu}$ where

$$
\begin{aligned}
L^{\mu\nu} &= 4[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - g^{\mu\nu} p_1 \cdot p_3)] \;, &(8)\\
H^{\mu\nu} &= 4[p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + g^{\mu\nu}(M^2 - p_2 \cdot p_4)] \;, &(9)
\end{aligned}
$$

$q^2 = (p_1 - p_3)^2$, and where M is the proton mass and the electron mass has been neglected.

Exercise 3 [4 marks]: By contracting the two tensors $L_{\mu\nu}$ and $H^{\mu\nu}$, show that the spin averaged amplitude squared becomes

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{8e^4}{((q^2))^2}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)M^2\right] .$$

Exercise 4 [2 marks]: Using the lab-frame four-momenta above and the definition of the Mandelstam variable t, show that $t = q^2 = -2E_1E_3(1-\cos\theta)$

Exercise 5 [10 marks]: By using four-momentum conservation ($p_1 - p_3 = p_4 - p_2$) show that in the lab frame $t = -2M(E_1 - E_3)$ and therefore that the outgoing electron energy is related to the scattering angle θ by

$$E_3 = \frac{E_1 M}{M + E_1(1-\cos\theta)} \tag{10}$$

Exercise 6 [12 marks]: Using the relations above and

$$\frac{d\sigma}{d\cos\theta} = \frac{d\sigma}{dt}\frac{dt}{dE_3}\frac{dE_3}{d\cos\theta} ,$$

find the differential cross section $d\sigma/d\cos\theta$ in terms of E_1, $\cos\theta$ and $\langle|\mathcal{M}|^2\rangle$. Using the solid angle element (for no dependence on ϕ) $d\Omega = 2\pi\, d\cos\theta$, thus show that

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2[M + E_1(1-\cos\theta)]^2} . \tag{11}$$

Exercise 7 [8 marks]: Using the lab-frame four-momenta from above, show that the spin-averaged amplitude squared becomes

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{(q^2)^2} M E_1 E_3 \left[(E_1 - E_3)(1-\cos\theta) + M(1+\cos\theta)\right] . \tag{12}$$

The combination of Eqs. (10), (11) and (12) gives the spin-averaged cross section for elastic electron-proton scattering including effects of spin and proton recoil.

Bonus question [0 marks]: Show that for $E_1 \ll M$ that $E_1 \approx E_3$ and that the formula for $d\sigma/d\Omega$ reduces to the *Mott* cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2\cos^2\frac{\theta}{2}}{4E_1^2\sin^4\frac{\theta}{2}} .$$

G. Cowan