PH4442 Advanced Particle Physics
Problem sheet 4
Due Monday, 1 December 2025

For the following exercises, consider the scattering reaction $\nu_\mu e^- \to \mu^- \nu_e$. Label the four-momenta and spins:

incoming ν_μ	p_1, s_1
incoming electron	p_2, s_2
outgoing μ^-	p_3, s_3
outgoing ν_e	p_4, s_4

Exercise 1 [4 marks]: Draw the lowest-order Feynman diagram for the reaction, labeling the particles, their momenta and spins.

Exercise 2 [4 marks]: Using $q = p_1 - p_3$, write down the invariant amplitude $\mathcal{M}$ in terms of the weak couplings g and the W mass M_W, and indicate what terms correspond to what parts of the diagram.

Exercise 3 [2 marks]: Show that for $q^2 \ll M_{\mathrm{W}}^2$, the amplitude (up to an arbitrary phase) is

$$\mathcal{M} = \frac{G_{\mathrm{F}}}{\sqrt{2}} \left[\overline{u}(p_3, s_3)\gamma_\mu(1-\gamma^5)u(p_1, s_1)\right] \left[\overline{u}(p_4, s_4)\gamma^\mu(1-\gamma^5)u(p_2, s_2)\right] .$$

Exercise 4 [10 marks]: The spin-averaged amplitude squared can be expressed as

$$\langle|\mathcal{M}|^2\rangle = \frac{G_{\mathrm{F}}^2}{2} M_{\mu\nu} E^{\mu\nu} ,$$

where the tensors M and E are defined as

$$\begin{aligned} M_{\mu\nu} &= \sum_{s_1,s_3} \left[\overline{u}(p_3, s_3)\gamma_\mu(1-\gamma^5)u(p_1, s_1)\right] \left[\overline{u}(p_3, s_3)\gamma_\nu(1-\gamma^5)u(p_1, s_1)\right]^* , \\ E^{\mu\nu} &= \frac{1}{2}\sum_{s_2,s_4} \left[\overline{u}(p_4, s_4)\gamma^\mu(1-\gamma^5)u(p_2, s_2)\right] \left[\overline{u}(p_4, s_4)\gamma^\nu(1-\gamma^5)u(p_2, s_2)\right]^* \end{aligned}$$

Notice here the tensor M does not have a factor of 1/2 as we suppose the incident ν_μ beam only contains left-handed neutrinos. In the tensor E, however, both spin states for the electron are present in the average. Neglecting the mass of the electron but not that of the muon, show that the tensors M and E are

$$\begin{aligned} M_{\mu\nu} &= 2\mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\not{p}_1\gamma_\nu(\not{p}_3 + m_\mu)\right] = 2\mathrm{Tr}\left[\gamma_\mu\not{p}_1\gamma_\nu\not{p}_3\right] + 2\mathrm{Tr}\left[\gamma^5\gamma_\mu\not{p}_1\gamma_\nu\not{p}_3\right] , \\ E^{\mu\nu} &= \mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)\not{p}_2\gamma^\nu\not{p}_4\right] = \mathrm{Tr}\left[\gamma^\mu\not{p}_2\gamma^\nu\not{p}_4\right] + \mathrm{Tr}\left[\gamma^5\gamma^\mu\not{p}_2\gamma^\nu\not{p}_4\right] . \end{aligned}$$

Hints: use $\{\gamma^\mu, \gamma^5\} = 0$ for all μ; the trace of an odd number of gamma matrices is zero (note $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is four gamma matrices); also recall $P_{\mathrm{L}}^2 = P_{\mathrm{L}}$ where $P_{\mathrm{L}} = \frac{1}{2}(1-\gamma^5)$ is the left-handed projection operator.

Exercise 5 [10 marks]: We will use the following trace theorems:

(i) $\mathrm{Tr}(\gamma^5 \not{a} \not{b}) = 0$,

(ii) $\mathrm{Tr}(\gamma^5 \not{a} \not{b} \not{c}) = 0$,

(iii) $\mathrm{Tr}(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = -4i\epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta = -4i\epsilon^{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta$,

where $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor: $\epsilon^{0123} = 1$, it is 1 for all cyclic permutations, -1 under exchange of adjacent indices, and 0 if any two indices are equal. Note with lowered indices one has $\epsilon_{0123} = -1$. Show that

$$\begin{aligned} M_{\mu\nu} &= 8\left([p_{1,\mu}p_{3,\nu} + p_{3,\mu}p_{1,\nu} - (p_1 \cdot p_3)g_{\mu\nu}] - i\epsilon_{\mu\alpha\nu\beta}p_1^\alpha p_3^\beta\right) , \\ E^{\mu\nu} &= 4\left([p_2^\mu p_4^\nu + p_4^\mu p_2^\nu - (p_2 \cdot p_4)g^{\mu\nu}] - i\epsilon^{\mu\lambda\nu\sigma}p_{2,\lambda}p_{4,\sigma}\right) . \end{aligned}$$

Show that the term in square brackets contracted with the ε term gives zero and the contraction of the bracketed terms with each other is

$$[\,]_{\mu\nu}[\,]^{\mu\nu} = 2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) .$$

You may then use the fact that the contraction of both ε terms after some calculation gives

$$\epsilon \cdot \epsilon = 2(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_1 \cdot p_4)(p_2 \cdot p_3) .$$

Exercise 6 [8 marks]: Using the results from above show that the spin-averaged amplitude squared is

$$\langle|\mathcal{M}|^2\rangle = 16G_{\mathrm{F}}^2 s(s - m_\mu^2)$$

Exercise 7 [4 marks] Find the differential cross section $d\sigma/d\Omega$ in the centre-of-mass frame.

Exercise 8 [4 marks] Show that in the limit $s \gg m_\mu^2$ that the total cross section is

$$\sigma(\nu_\mu e^- \to \mu^- \nu_e) = \frac{G_{\mathrm{F}}^2}{\pi} s .$$

Exercise 9 [4 marks] Find the total cross section in the lab frame where the initial electron is at rest and show that it is proportional to the incident neutrino energy E_{ν_μ}.

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