

Exercise 1 [10 marks]: Consider (as in Problem Sheet 1) the joint pdf for the continuous random variables x and y

$$f(x, y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2, \\ 0 & \text{otherwise.} \end{cases}$$

Define the new variables

$$\begin{aligned} u &= \sqrt{x^2 + y^2}, \\ v &= \tan^{-1}(y/x). \end{aligned}$$

That is, u corresponds to the radius and v to the azimuthal angle in plane polar coordinates, with $u \geq 0$ and $0 \leq v < 2\pi$.

(a) [5] Find the joint pdf of u and v . (Use the inverse of the transformation $x = u \cos v$, $y = u \sin v$.) Are u and v independent? Justify your answer.

(b) [5] Find the marginal pdfs for u and v .

Exercise 2 [5 marks]: Consider n random variables $\vec{x} = (x_1, \dots, x_n)$ that follow a joint pdf $f(\vec{x})$ and constants $c_0, c_1, \dots, c_n$.

(a) [1 mark] Starting from the definition of the expectation value for continuous random variables, show that

$$E \left[c_0 + \sum_{i=1}^n c_i x_i \right] = c_0 + \sum_{i=1}^n c_i E[x_i].$$

(b) [4 marks] Using the result from (a), show that the variance is

$$V \left[c_0 + \sum_{i=1}^n c_i x_i \right] = \sum_{i,j=1}^n c_i c_j \text{cov}[x_i, x_j].$$

For the variance above, find what this reduces to in the case where the variables $x_1, \dots, x_n$ are uncorrelated.

Exercise 3 [5 marks]: Consider two random variables x and y and a constant α . From the previous exercise we have (no need to rederive)

$$V[\alpha x + y] = \alpha^2 V[x] + V[y] + 2\alpha \text{cov}[x, y] = \alpha^2 \sigma_x^2 + \sigma_y^2 + 2\alpha \rho \sigma_x \sigma_y,$$

where $\sigma_x^2 = V[x]$, $\sigma_y^2 = V[y]$, and the correlation coefficient is $\rho = \text{cov}[x, y] / \sigma_x \sigma_y$. Using this result, show that the correlation coefficient always lies in the range $-1 \leq \rho \leq 1$. (Use the fact that the variance $V[\alpha x + y]$ is always greater than or equal to zero and consider the cases $\alpha = \pm \sigma_y / \sigma_x$.)